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NOTES ON MECHANISM

by

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Reference is made herein to the textbook Elements of Mechanism, Fifth Edition, by Schwamb, Merrill and James.

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NOTES ON MECHANISM

I. Fundamental Principles

A. Displacement

1. *Complanar Displacement.*--The definition of displacement requires the use of certain terms which themselves would be unintelligible without definition. These terms must therefore be defined.

Definition I.--*Complanar Figure.* A figure is said to be *complanar* when all the lines (whether straight or curved) and points of which it is composed lie in the same plane.

Comments.--1. Any figure whether plane or of three dimensions may be considered to be made up of points and lines. This is axiomatic.

2. It is, of course, not necessary for a figure to be at rest to be complanar. It may move either in its own plane or in any other possible way without affecting its planeness. Let the student think of a cube of steel, with six perfectly plane square faces, hurling through space. Each of the faces obviously remains plane despite the motion.

Definition II.--*Congruent Complanar Figures.*--Two complanar figures are said to be *congruent* when similar and equal and so situated relative to each other in the same plane that they can be brought into coincidence without removing either figure from the plane and turning it upside down.

Comments.--1. This definition of congruency is somewhat narrower than that of the geometer, but it will serve our purpose just as well as or perhaps even better than the geometer's definition.

2. Perhaps a figure will make this idea of congruency somewhat clearer.

In Fig. 1 triangle $A_2B_2C_2$ is constructed similar and equal to triangle $A_1B_1C_1$, and obviously either can be moved into coincidence with the other without taking either one out of the plane and turning it upside down. They are thus congruent.

Triangles $A_1B_1C_1$ and $A_2B'_2C_2$ are also similar and equal by construction but clearly neither can be brought into coincidence with the other without removing one or the other from the plane and turning it upside down. They are thus not congruent.

The geometer says that triangles $A_1B_1C_1$ and $A_2B_2C_2$ are *directly* congruent and triangles $A_1B_1C_1$ and $A_2B'_2C_2$ *indirectly* congruent.

For the sake of simplicity we have used triangles to illustrate the meaning of this term, but obviously what has been said would apply without change to complanar figures of the most general kind.

The student is strongly advised to consult the unabridged dictionaries concerning this term and all such terms. Webster's New International Dictionary has a particularly good and scientific discussion of the word in question.

Definition III.--*Collinear.* Three or more points are said to be *collinear* when they lie in the same straight line.

Comment.--When three or more points, in obedience to the geometry of the figure in which they occur, and not just fortuitously, fall on the same straight line, it is a matter of important significance, and the student should be sure not to pass it by without investigation.

Many cases of collinearity occur in the geometrical figures with which the student of mechanism is concerned and every one of them has important kinematic significance.

Definition IV.--*Rigid Body:* A piece of imaginary matter the distance between no two points of which would change however large the forces acting on the body might be.

Comment.--Of course there is no such matter--and for this reason it is said to be imaginary,--but there is no reason why we should not endow matter with this imaginary property if to do so serves a useful purpose. And it does serve a useful purpose to make this hypothesis because none of the kinematic theorems with which we shall deal presently is strictly true unless the bodies with which they deal are perfectly rigid.

The bodies of which our machines are made are far from being rigid; indeed they behave much more like indiarubber than they do like rigid bodies; but they are sufficiently rigid to serve all practical purposes; in fact, they are better for machine purposes as they are than they would be if they were perfectly rigid.

The bodies of which our mechines are made are not only sufficiently rigid to enable them to serve well all practical purposes, but they are so nearly rigid that the kinematic theorems presently to be discussed apply to them without discernible error.

Definition V.--*Instant:* An infinitely short period of time.

Comment.--For our purposes an instant is not a moment or a second or even as much as the millionth or billionth part of a moment or a second; it is a portion of time infinitely shorter than even the smallest of these. It is a portion of time so short that a rigid body traveling at the rate of a billion miles a second, for example, would not move a distance as great as the billionth part of the thickness of this paper in an instant. It is a portion of time represented in the differential calculus by the symbol dt, in which t represents time.

Definition VI.--*Configuration.* The geometric figure formed at and for a given instant by a system of stationary or moving rigid bodies, each of which sustains to each of the others at the instant a known position, is called the configuration of the system.

Comments.--1. This quite indispensable term is in constant use by both the teacher and the student of mechanism and the student must therefore master its meaning before he has been in class two days. If he fails to do this his progress in the course will be seriously interfered with if not entirely arrested.

2. The meaning of this definition can be made perfectly clear by illustrating it with the disc-plane device. Let the student imagine a perfect plane surface formed on a hardened slab or plate of steel. To fix his ideas he may give the slab the following dimensions: length, 6 feet; width, 4 feet; thickness, 3 inches. The plane surface formed on the slab of steel is 4' × 6'. Now let the student imagine also a number (say five) of round, flat, steel discs, 1" thick and 4" in diameter. Each disc, let him imagine, has two perfect, parallel, plane surfaces formed on it. Put the slab in a horizontal position with its plane surface up, and lay the discs on it. Number the slab with the number 1, and the discs with the numbers 2, 3, 4, 5, and 6 respectively. Then let the student further imagine: (1) that at and for a given instant the discs, keeping always in plane contact with the plane surface on the slab, move with any speed along any paths, that is, that would prevent the discs from fouling each other; (2) that an infinitely quick and perfect camera is placed above and at a known distance from the slab, with its axis perpendicular to the plane surface of the slab. Then the picture of the moving discs that this camera would give at the instant would be the *picture of the configuration* of the system of discs at the instant.

From such a picture, if we knew its scale, the following information could be obtained: (1) the distance from any point in any one of the discs to any point in any other one, (2) the angle between any straight line scribed on the top plane surface of any one of

the discs and any straight line scribed on the top plane surface of any other disc. Obviously the word *top* in (2) of the preceding sentence may be changed to *bottom* without affecting the truth of the statement. Obviously, moreover, if we imagine a plane, Q_1 let us call it, parallel to and at a distance of less than 1" from the plane surface formed on the slab, this plane Q_1 will cut from each disc a circle, and, as the discs move under the circumstances specified, these circles will remain in plane Q_1. Let us imagine, in the next place, the planes of these circles to be of indefinite extent and fastened to their respective discs, and let us call these coincident planes Q_2, Q_3, Q_4, Q_5, and Q_6 respectively. Then the photographic *picture or configuration* of the system of discs will give us: (1) the angle between any straight line in any one of these five planes--Q_2, Q_3, Q_4, Q_5, Q_6--and any straight line in any other one of them; (2) the distance from any point in any one of these planes to any point in any other one of them.

3. In reading what has just been written about the term *configuration* the student must keep in mind that, unless the contrary is specifically stated, both the straight line and the plane are to be thought of as being unlimited in extent. He must remember, too, that two straight lines in the same plane-- that is two complanar lines--always intersect; even when they are parallel they intersect at infinity.

4. The student should notice that the discs used to illustrate the definition of the word *configuration* need not be circular nor of any given shape or thickness. All that is essential is that they be rigid and possess at least one plane surface.

We are now prepared to define the term, *complanar displacement.*

Definition VII.--*Complanar Displacement of a Rigid Body.* When one rigid body moves relative to a second from one position to any other in such a way that a plane section of the second body remains throughout the motion in a plane fixed relative to the first body, the second body has undergone *complanar displacement* relative to the first and the first has undergone the same *complanar displacement* in the opposite sense relative to the second.

Comments.--1. The time in which the displacement takes place doesn't enter into the question; that is, the relative displacement of the two bodies is the same whether it takes place in the billionth part of a second or in a billion years. Thus questions of displacement are purely geometric. And the student who hopes to learn something about the science of mechanism had better begin at once to review his plane geometry.

2. The definition of the term may be illustrated with the device used to illustrate Definition VI. Anyone of the discs in that device, say disc 2, undergoes *complanar* displacement relative to slab 1 if it is moved from any given position to any other, because obviously plane Q_2 will remain in coincidence with plane Q_1 throughout the movement.

3. The displacement may be, as will appear later, either (1) *angular* or (2) *translatory.* In the first case, the points of disc 2 that lie in plane Q_2 may, as we shall see, travel along concentric circles of finite radii as disc 2 moves from its first to its second position; in the second case, these circles become parallel straight lines. Although this states an interesting and useful possibility, the points of disc 2 that lie in plane Q_2 may move along an infinite number of different paths as the disc undergoes its displacement. We shall discuss this particular question in more detail later, in connection with Theorem II.

4. The concept of *complanar displacement* is of the utmost importance to the student of mechanism because the links or members of most machines undergo, as the machines function, *complanar displacement* relative to each other.

The Kinematic Theorems of Complanar Displacement

If one's study of mechanism is to be successful it is necessary for one to become thoroughly familiar with the fundamental kinematic theorems that are now to be considered.

Without a knowledge of these theorems, let the student be assured, no one can really understand the science of mechanism, because these theorems bear to the subject of mechanism about the same relation that the multiplication table does to the science of mathematics.

Theorem I.--The *complanar displacement* of one rigid body relative to a second rigid body is completely determined by the complanar displacements of any two points of the body that lie in the plane of displacement.

Proof.--Let contour 1, Fig. 2, represent a plane section of rigid body 1, let contour 2_1 represent the same plane section of rigid body 2, let P_2 and Q_2 represent any two points in body 2 in the plane of the section, and let P_2' and Q_2' be the positions that points P_2 and Q_2 take after body 2 has been displaced complanarily from position 2_1 to any other position 2_2. Then we are to show that we can find the position to which *any* third point in body 2 displaces when we know that points P_2 and Q_2 displace to P_2' and Q_2', respectively.

1. Take, first, a point, R_2, anywhere in contour 2_1. On $P_2'Q_2'$ construct triangle $P_2'Q_2'R_2'$ *congruent* to triangle $P_2Q_2R_2$; then obviously R_2', thus determined, is, since body 2 is rigid, the position to which R_2 displaces when points P_2 and Q_2 are simultaneously displaced to P_2' and Q_2', respectively. Clearly, moreover, the position to which any second point in the plane of displacement displaces, whether on contour 2_1 or not, can be found by exactly the same construction.

2. Take, secondly, a point S_2 *anywhere* in body 2. On $P_2'Q_2'R_2'$ as a base construct tetrahedron $P_2'Q_2'R_2'S_2'$ congruent to tetrahedron $P_2Q_2R_2S_2$; then point S_2' so determined fixes the position to which point S_2 goes when P_2 goes to P_2' and Q_2 to Q_2'.

Comments.--1. Thus we see that when we know for complanar motion the displacement of any two points of a rigid body that lie in the plane of displacement we know enough to determine the displacements of all points of the body, that is to say, we know enough to determine the displacement of the body itself.

2. This theorem is very useful to us because it allows us in all complanar kinematic questions to substitute two points or a straight line for a rigid body.

Theorem II.--Any complanar displacement of a rigid body can always be reduced to a single rotation.

Proof.--In accordance with Theorem I let the two straight lines P_2Q_2 and $P_2'Q_2'$, Fig. 3, represent two different positions of rigid body 2, which has suffered a complanar displacement relative to body 1. Then we have to show that body 2 can be displaced from its first to its second position by a single rotation.

Three cases arise, because the second position of the rigid body may bear to its first position any one of three relations, of which one is general and the other two special.

Case I.--For this case, the general case, the quadrilateral $P_2Q_2Q_2'P_2'$ is general.

At the middle points, E and F, of lines P_2P_2' and Q_2Q_2', Fig. 3, erect perpendiculars and find their point of intersection *12*. Join this point to Q_2, P_2, P_2', and Q_2', thus forming the two pairs of triangles *(12)* P_2P_2' and *(12)* Q_2Q_2 and *(12)* P_2Q_2 and *(12)* $P_2'Q_2'$. The first two triangles are evidently isosceles, because the lines *(12)* E and *(12)* F are the perpendicular bisectors of the segments P_2P_2' and Q_2Q_2', respectively. But if triangles *(12)*P_2P_2' and *(12)*Q_2Q_2' are isosceles, triangles *(12)*P_2Q_2 and *(12)*$P_2'Q_2'$ are evidently congruent. And if triangles *(12)*P_2Q_2 and *(12)*$P_2'Q_2'$ are congruent, either can be brought into coincidence with the other by a single rotation about the straight line, called the displacement axis, that passes through point *12*, the vertex that these two triangles have in common, perpendicular to the plane of displacement And this means, of course, that body 2 can be displaced from position 2_1 to position 2_2 by a single rotation about displacement axis *12*.

Comments on Case I.--This theorem plays a very important part in the proof of Theorem VI, one of the most useful theorems in the science of mechanism; the present theorem must, therefore, be carefully and fully considered.

1. When body 2 rotates about displacement axis *12* in going from position 2_1 to position 2_2 by a single rotation all points in it describe circles whose centers are in axis *12* and whose planes are perpendicular thereto. Thus points P_2, Q_2, R_2 of body 2, in the plane of displacement, describe the concentric circular arcs, shown in the figure, whose center is *12*.

2. Since body 2 is rigid, all straight lines of it in the plane of displacement must turn through the same angle during the *complanar* displacement of body 2 from position 2_1 to position 2_2. This can easily be shown by very elementary Euclidean geometry.

Proof.--Take, for example, lines $P_2(12)$ and P_2R_2 and suppose them to be fastened to body 2. When body 2 is displaced complanarily from position 2_1 to position 2_2 line $(12)P_2$ goes, as we have seen, to $(12)P_2'$ and line P_2R_2 to $P_2'R_2'$. Thus we are to show that the acute angle β_1 between $(12)P_2$ and $(12)P_2'$ is equal to the acute angle α_1 between P_2R_2 and $P_2'R_2'$, for these, evidently, are the angles turned through by lines $(12)P_2$ and P_2R_2, respectively, during the displacement of body 2.

Now by hypothesis triangles $P_2Q_2R_2$ and $P_2'Q_2'R_2'$ are congruent and we have shown that triangles $P_2Q_2(12)$ and $P_2'Q_2'(12)$ are congruent; thus it follows that angles α_2 and β_2 are equal. But if these two angles are equal triangles P_2TG and $P_2'T(12)$ are similar, and thus angles α_1 and β_1 are equal, which is what was to be proved.

By exactly the same reasoning it can be shown that any other line fastened to body 2 turns through an angle equal to β_1 during the specified complanar displacement of body 2.

3. The student should be sure to notice and prove that triangles $P_2P_2'(12)$, $Q_2Q_2'(12)$, and $R_2R_2'(12)$ are similar isosceles triangles.

Case II of Theorem II.--For this case, the first of the two special cases of this theorem, the second position of rigid body 2 bears to the first position such a relation that quadrilateral $P_2P_2'Q_2'Q_2$ is a parallelogram, Fig. 4.

Comments.--1. Since by hypothesis figure $P_2P_2'Q_2'Q_2$ is a parallelogram, sides P_2P_2' and Q_2Q_2' are parallel and thus also their perpendicular bisectors $E(12)$ and $F(12)$; that is, these perpendiculars meet at infinity. In this case, therefore, displacement center *12* is at infinity, and body 2 in order to move from position 2_1 to position 2_2 by a single rotation must turn about the axis that passes through this point *12* at infinity perpendicular to the plane of displacement; in other words, to go from position 2_1 to position 2_2 by a single rotation the points of body 2 must move in parallel straight lines of the same length. Now when a rigid body moves in this way the displacement that it undergoes is called *rectilinear translation*.

2. The student should be sure to get the analogy between this case and the preceding one. Both displacements are effected by a single rotation, but in the first case the axis of displacement is at a finite distance, whereas in the second case it is at infinity. In both cases all the points of body 2 describe circles in body 1, but in the first case the radii of these circles are finite in length, whereas in the second case they are infinite in length. In both cases all lines of body 2 in the plane of displacement turn through the same angle during the displacement, but in the first case this angle has some finite value greater than zero, whereas in the second case it is zero.

3. The most important thing of all to notice is that Case II does not constitute an exception to the truth of Theorem II; in other words, that there is, even in this highly special case, when the two positions of the rigid body are parallel, an axis about which body 2 can move by a single rotation from the one position to the other.

Case III of Theorem II.-- For this case, the special case of Theorem II, the second position of body 2 is such that quadrilateral $P_2P_2'Q_2'Q_2$, Fig. 3, is an isosceles trapezoid, Fig. 5.

Comment.--For this case by hypothesis sides P_2P_2' and Q_2Q_2' of the quadrilateral $P_2P_2'Q_2'Q_2$ are parallel but not of the same length, while sides P_2Q_2 and $P_2'Q_2'$ are of the same length and cannot, therefore, be parallel; thus it follows (1) that $P_2P_2'Q_2'Q_2$ is an isosceles trapezoid and (2) that perpendicular bisectors *E(12)* and *F(12)* coincide, and therefore have no point of intersection. The construction on which we have depended in the other two cases for locating displacement center *12* therefore fails in this case. But this is of no consequence since, obviously, center *12* is here the point of intersection of lines P_2Q_2 and $P_2'Q_2'$.

Comment on Case III of Theorem II.--Everything that was said concerning Case I of this theorem is applicable here with only slight and obvious modification.

General Comments on Theorem II.--1. For each of these three cases, therefore, Theorem II holds, and since these are the only cases that can arise, it follows that Theorem II is true for all possible cases.

2. It should be particularly noticed that in each of these three cases displacement axis *12* is a line in both bodies, 1 and 2, during the displacement. This is so because it evidently does not change its position relative to either of these bodies during the displacement. *This is a very fruitful idea and the student will do well to make careful note of it.*

Theorem III.--Any complanar displacement of a rigid body can be resolved in an infinite number of ways into two component, complanar displacements, both of which may be angular, or one of which may be angular and the other rectilinear.

Proof.--Case I. Let body 2, Fig. 6, be displaced as follows: First, from position 2_1 to any other position 2_2 by a rotation about any displacement axis *12*, chosen at random, and let this be the given displacement; secondly, let body 2 be displaced from position 2_1 to any other position 2_3 by a rotation about any arbitrarily chosen displacement axis *12"*. Then, Theorem II, the body can be displaced from position 2_3 to position 2_2 by a single rotation about displacement axis *12'*.

Thus we see that the given complanar displacement 2_1——2_2 can be resolved into the two component complanar displacements 2_1——2_3 and 2_3——2_2. Moreover, since component displacement 2_1——2_3 was chosen perfectly arbitrarily and since both it and displacement 2_3——2_2 are angular, it follows that the given displacement 2_1——2_2 can be decomposed into two other angular complanar displacements in an infinite number of ways.

Case II. Let 2_1——2_2, Fig. 7, be the given complanar displacement of body 2. Then give this body the following two successive complanar displacements: (1) the arbitrary rectilinear displacement 2_2——2_3, in any direction and of any amount about axis *12'* at infinity, as shown; (2) by Theorem II, the angular displacement 2_3——2_1 about displacement axis *12"*. Thus it is evident that complanar displacement 2_1——2_2 can be resolved into the two component displacements 2_2——2_3 and 2_3——2_1, the former being rectilinear and the latter angular. Moreover, since 2_2——2_3 was chosen arbitrarily, it follows that the resolution can take place in an infinite number of ways.

Or we may proceed in the following way to decompose angular displacement $\alpha_1 = 2_1$——2_2 into an angular component 2_1—— $2_3 = \alpha_1$ and a rectilinear component 2_3----- 2_2. Choose a displacement center *12"* anywhere and rotate body 2 about this center from position 2_1 through the angle α_1 into position 2_3; then by rectilinear translation displace body 2 from position 2_3 to position 2_2.

Comments on Theorem III.--1. Having resolved a given complanar displacement 2_1——2_2 into two component complanar displacements 2_1——2_3 and 2_3——2_2, we can by the same

process, evidently, resolve each of these components into two others, thus resolving the given displacement into four components. Then each of these four components can be resolved into two others, and so on, *ad infinitum*. Thus it follows that a given displacement can be resolved into any number of components.

2. Curvelinear Translation.--Since we can, as is now evident, resolve rectilinear displacement 2_2———2_3, Fig. 7, into an infinite number of other rectilinear displacements in an infinite number of ways, it follows that body 2 can be displaced from position 2_2 to position 2_3 by moving any point of it along any arbitrarily chosen complanar curve we please, say for example, by moving point P_2' of it along curve $P_2' \ldots P_2''$. Moreover, whatever kind of complanar curve $P_2' \ldots P_2''$ may be, any other point of body 2 will, it ought to be quite obvious by now, describe a curve exactly like it. (If the student has any trouble seeing the truth of this last statement it is because his geometrical experience is entirely too limited for the successful pursuit of a course in Mechanism.)

Theorem IV.--If three rigid bodies, 1, 2, 3, undergo simultaneous, angular, complanar, relative displacements, these three displacements are connected by the equation

$$D_{31} = D_{21} + D_{32}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Unless, however, we impose the axiomatic conditions expressed by the three equations

$$\left.\begin{aligned} D_{31} &= -D_{13}, \\ D_{21} &= -D_{12}, \\ D_{23} &= -D_{32} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

equation (1) would not hold for all possible cases. Each of these axiomatic expressions asserts only what is perfectly obvious, namely, that if rigid body x turns through an angle D_{xy} relative to rigid body y, then rigid body y turns through an angle of the same size in the opposite sense relative to body x. In other words, the truth expressed by equations (2) is axiomatic.

Proof of Theorem IV.--In Fig. 8 three rigid bodies, 1,2,3, are shown. The three complanar lines L_1, L_2, L_3, which appear in the figure, are attached to bodies 1,2,3, respectively; so that when one of these rigid bodies turns through a certain angle relative to a second, the line attached to the first body turns through the same angle relative to the line attached to the second body. Before displacement took place lines L_1, L_2, and L_3, it must be assumed, coincided with each other; after displacement they form the triangle *EFG*, shown in the figure; during displacement line L_2 (or body 2) turned counterclockwise through angle D_{21} from coincidence with line L_1, line L_3 (or body 3) turned counterclockwise through angle D_{31} from coincidence with line L_1, and line L_3 turned counterclockwise through angle D_{32} from coincidence with line L_2.

The truth of the theorem for the conditions shown in the figure is now at once evident, because by Euclid, Book I, Proposition XVIII, the exterior angle D_{31} of triangle *EFG* is equal to the sum of the two opposite interior angles D_{21} and D_{32}; that is to say, as was to be proved,

$$D_{31} = D_{21} + D_{32}.$$

But, it should be observed, we are not entitled to regard this as a general proof of Theorem IV because any one of the three bodies can turn either clockwise or counterclockwise relative to each of the other bodies to form the triangle *EFG* of given shape. Let us, therefore, work out a more general proof of this theorem.

Choose a point *O* anywhere in the plane of displacement, Fig. 8a, and through it draw lines parallel to L_1, L_2, L_3, respectively. Then the angles subtended at *O* by the arcs *AB*, *BC*, and *AC* are equal to the angular displacements D_{21}, D_{32}, D_{31}, respectively, of bodies 1, 2,3. And as triangle *EFG* changes its shape, that is, as bodies 1,2,3, undergo complanar angular displacement in all possible ways relative to each other, points *A*, *B*, and *C* will assume all possible positions, and the angles D_{21}, D_{31}, D_{23} all possible values. Let the

circle, Fig. 8a, swept from O as a center, be of unit radius; then the number of linear units in the arc AB will be equal to the number of radians in the angle, D_{21}, subtended by arc AB. Similarly for arcs AC and BC; that is to say, arcs AB, BC, and AC will numerically equal angles D_{21}, D_{32}, and D_{31}, respectively. The six possible relative positions of points A, B, C on the circle of unit radius are shown in Fig. 9. For the first two of these figures point B lies between A and C, for the next two point C lies between A and B, for the last two point A lies between B and C; in other words, for diagrams 9(a) and 9(b) line 2 lies between lines 1 and 3, for diagrams 9(c) and 9(d) line 3 lies between lines 1 and 2, and for diagrams 9(e) and 9(f) line 1 lies between lines 2 and 3. That these six arrangements exhaust all of the possibilities, even a very cursory consideration of Fig. 9 will make perfectly evident.

Moreover, first, for arrangements (a) and (b), Fig. 9,

$$arc\ AC = arc\ AB + arc\ BC,$$

that is--since arcs and angles are numerically equal--

$$D_{31} = D_{21} + D_{32}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

secondly, for arrangements (c) and (d), similarly,

$$arc\ AB = arc\ AC + arc\ CB,$$

that is

$$D_{21} = D_{31} + D_{23} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

thirdly, for arrangements (e) and (f)

$$arc\ BC = arc\ BA + arc\ AC,$$

that is

$$D_{32} = D_{12} + D_{31} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Now by using equations (2) and the law of transposition in algebra it is easy to transform either (4) or (5) into equation (3). Thus we conclude that if we take into account sense of displacement, Theorem IV is true for all possible cases of relative, complanar, angular displacement.

Comments on Theorem IV.--1. There are three other ways in which, by the help of equations (2) this theorem may be expressed symbolically in the form of equation (3), that is, with one term on the left-hand side of the equation and all three terms positive. These forms are:

$$D_{13} = D_{23} + D_{12}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

$$D_{12} = D_{32} + D_{13}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

$$D_{23} = D_{13} + D_{21}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

The student must not, of course, think that equations (3), (4), (5), (6), (7), and (8) are independent of each other because, as a matter of fact, the entire six give us just exactly as much as but no more information about the six displacements in question than any one of the six does.

In most problems involving Theorem IV some particular one of these equations will be more convenient than any of the others. That is the reason why we have called attention to all six forms.

2. The student should be sure to note the rule that controls the order of the subscripts in these equations. Take equation (3), for example. This rule, which consists of four parts, is as follows: (1) The first numeral in the subscript of the first term on the right-hand side of the equation is that one of the three numerals that is absent from

the subscript of the term on the left-hand side of the equation; (2) the second numeral of the subscript of the first term on the right-hand side of the equation is the same as the second numeral of the subscript of the term on the left-hand side of the equation; (3) the first numeral of the subscript of the second term on the right-hand side is the same as the first numeral of the subscript of the term on the left of the equation; (4) the second numeral of the subscript of the second term is the same as the first numeral of the subscript of the first term on the right.

If the student will take the trouble to examine equations (4), (5), (6), (7), and (8), he will find that the rule just stated is true for each of them.

Theorem V.--If of the three rigid bodies, 1,2,3, that have undergone complanar displacement, body $\begin{Bmatrix}1\\2\\3\end{Bmatrix}$ has undergone a complanar translatory displacement relative to body $\begin{Bmatrix}3\\1\\2\end{Bmatrix}$ and body $\begin{Bmatrix}2\\3\\1\end{Bmatrix}$ has undergone a complanar, translatory displacement relative to body $\begin{Bmatrix}1\\2\\3\end{Bmatrix}$, then body $\begin{Bmatrix}2\\3\\1\end{Bmatrix}$ has undergone a complanar, translatory displacement relative to both $\begin{Bmatrix}3\\1\\2\end{Bmatrix}$ whose magnitude is the vector sum of the other two displacements.

Proof.--1. That the resulting displacement in any one of the three cases must be translatory it is easy to see. Take, for example, the second case, the case, that is, that corresponds to the horizontal row of numbers 2,1,3, 2,3,1 in the statement of the theorem. By hypothesis body 2 has had no angular displacement relative to body 1 and body 3 has had none relative to body 2; therefore, evidently, body 3 has had none relative to body 1; in other words, the displacement of body 3 relative to body 1 must, as the theorem asserts, have been translatory.

2. Let P_1, P_2, P_3, Fig. 10, be three coincident points in bodies 1,2,3, respectively. Let body 2 undergo a translatory, complanar displacement relative to body 1 such that point P_2 of body 2 travels along *any* curve α_{21} in body 1 to position ${}_3P_2$. Then, since body 2 has translatory displacement relative to body 1, any other point of body 2 travels along a curve exactly like and parallel to curve α_{21}. Now while P_2 is describing its path α_{21} in body 1 let point P_3 of body 3 describe any path α_{32}, in body 2. Then just as in the other case, any other point in body 3 will describe a path in body 2 exactly like and parallel to path α_{32}, because, by hypothesis, body 3 has translatory displacement relative to body 2. When, now, point P_2 has arrived at position ${}_3P_2$ relative to body 1 point P_3 will be at ${}_3P_3$ relative to body 2, so that the displacement of P_3 or, what is the same thing, the displacement of body 3 relative to body 1 is P_3——${}_3P_3$, that is, it is the third side of the triangle of which the other two sides are the two given displacements.

Thus it follows, evidently, that the displacement of body 3 relative to body 1 is, as the theorem asserts, the vector sum of the displacement of body 2 relative to body 1 and the displacement of body 3 relative to body 2. That is to say, symbolically,

$$L_{31} = L_{21} \leftrightarrow L_{32}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

in which L_{31} is the displacement of body 3 relative to body 1, L_{21} the displacement of body 2 relative to body 1, and L_{32} the displacement of body 3 relative to body 2.

The student must not, however, make the mistake of thinking that point L_3 in the second case, for example, will move along some particular path relative to body 1 in going from its first position to its final position ${}_3P_3$. It may be made to move along *any* path in body 1 that we may choose, such for instance as the path α_{31}, provided the law connecting the *given* displacements is what is required.

If the two given displacements take place simultaneously and are so related that for each position of point P_2 on its path α_{21} (in body 1) a certain position of point P_3 on its path α_{32} (in body 2) corresponds, then point P_3 will describe in body 1 some definite path α_{31}.

Again, if point P_2 (of body 2) moves along the straight-line path P_2——$_3P_2$ in body 1 in going from position P_2 to position $_3P_2$, and if at the same time point P_3 moves along the straight-line path $_3P_2$——$_3P_3$ in body 2 in going from position $_3P_2$ to position $_3P_3$, point P_3 (of body 3) will move along the straight line path P_3——$_3P_3$ in body 1, provided the displacement of P_2 relative to body 1, at all stages of the displacement, bears to the displacement of P_3 relative to body 2 the constant ratio $(P_3$——$_3P_2)/(_3P_2$——$_3P_3)$.

If, on the other hand, the two given displacements do not occur simultaneously but successively, then, obviously, the path described by point P_3 (of body 3) in body 1 is in the first case P_2, $_1P_2$, $_2P_2$, $_3P_2$, $_1P_3$, $_2P_3$, $_3P_3$ and in the second case the two straight line sides, P_2 $_3P_2$ and $_3P_2$ $_3P_3$, of the triangle P_2, $_3P_2$, $_3P_3$.

Problems of Complanar Displacement

Problem 1.--Of the lever-crank 4-link linkage, Fig. 11, given: (1) the lengths of the links 1, 2, 3, 4; (2) the angular displacement D_{21}; (3) some configuration of the linkage for example, the zero configuration 14, 12, 23_0, 34_0; required: (1) D_{41}, (2) D_{24}, (3) D_{23}, (4) D_{31}, (5) D_{34}, that is, the five displacements that arise when crank 2 undergoes the given displacement, D_{21}, relative to link 1.

Solution.--The total number of displacements for this 4-link mechanism is evidently $n(n - 1)/2$ where n is the number of links in the mechanism, because there is one displacement for each pair of links and the number of pairs of links is $n(n - 1)/2$. Since, moreover, n is four in this case, the total number of displacements is six, the six given in the statement of the problem.

The first configuration of the mechanism, called the zero configuration and designated with the symbol C_0, is fixed by the four points 14, 12, 23_0, 34_0; the other configuration, which we may designate with the symbol C_1, is given by the points 14, 12, 23_1, 34_1. The second configuration is found from configuration C_0 as follows: Locate point 23_1 on the crank-pin circle so that the angle between lines 12-23_0 and 12-23_1 is equal to D_{21}; from centers 23_1 and 14 with radii equal to 23_0-34_0 and 14-34_0, respectively, strike arcs, cutting each other in point 34_1.

1. Displacement D_{41}. This displacement is quite obviously the angle between the two lines 14-34_0 and 14-34_1; in other words, when crank 2 turns relative to link 1 from position 2_0 to position 2_1 lever 4 turns relative to link 1 from position 4_0 to position 4_1, through angle D_{41}.

If we draw through center 14, Fig. 11, the line L_4, making with line 14-12 the angle $D'_{41}(= D_{41})$, we thus obtain the line of link 4 that coincided with line 12-14 of link 1 before displacement took place, that is, for configuration C_0.

2. Displacement D_{24}. Consider triangle 14, A, 12, formed by lines 14-12, L_4, and 12-23_1. Angle $D'_{41}(= D_{41})$ of this triangle is, as we have just seen, the angular displacement of link 4 relative to link 1 that corresponds to the given counterclockwise angular displacement, D_{21}, of link 2 relative to link 1, and the interior angle at A between lines A-12 and A-14, measured counterclockwise from line A-14, is the angular displacement of link 2 relative to link 4, because for configuration C_0, these two lines, the first attached to link 2 and the second to link 4, coincided, so that the angle that they make for configuration C_1 is the relative angular displacement of the two links to which they are attached. Moreover, we see from the geometry of the figure that

$$D_{21} = D_{41} + D_{24},$$

which is the relation that Theorem IV tells us ought to hold for the three angular displacements of these three links, 1, 2, 4.

4. Displacement D_{23}. Consider Fig. 12, which shows the mechanism in the same two configurations as those of Fig. 11.

Through center 23_1 draw line L_3 so that the acute angle between it and line 23_1-34_1 is equal to the acute angle between 12-23_0 and 23_0-34_0; then L_3 is the line of link 3 that coincided with line 12-24 for configuration C_0; and the angle between L_3 and 12-23_1 is thus angular displacement D_{23}. 14

4. Displacement D_{31}. Since, Fig. 12, line L_3 is the line of link 3 that coincided with line 14-12 of link 1 when the mechanism was in configuration C_0, the angle between L_3 and 14-12 is displacement D_{31}. Moreover, the exterior angle D_{23} of triangle 12, 23_1, B is, it is evident, equal to the sum of the two interior angles D_{21} and D_{13}, a relation that connects the angular displacements of the three links 1, 2, 3, in accord with Theorem IV.

It is evident, too, that displacement D_{31} is also the angle between lines 23_0-34_0 and 23_1-34_1, and this fact, of course, leads to the most direct determination of displacement D_{31}.

5. Displacement D_{34}. Draw line L_3', Fig. 11, through point 34_1 making with line 23_1-34_1 the same obtuse angle that 14-34_0 makes with 23_0-34_0; then line L_3' is the line of link 3 that coincided with line 14-34_0 for configuration C_0, and thus the angle between L_3' and 14-34_1, read clockwise from 14-34_1, is displacement D_{34}.

Displacement D_{34} is also shown in Fig. 11 as the angle between lines L_3 and L_4. These two lines coincide with each other when the mechanism is in configuration C_0; hence the angle between them when the mechanism is in configuration C_1 is displacement D_{34}. From triangle 14, C, B, Fig. 11, we see, moreover, that

$$D_{43} = D_{13} + D_{41}$$

which is, of course, the relation that Theorem IV gives for these three displacements.

Problem 2.--Consider the lever-crank 4-link mechanism shown in Fig. 13. P_1 in this figure represents a pin fastened to link 1, the frame of the mechanism. This pin projects out into and beyond the plane of motion of lever 4; determine whether or not lever 4 will foul it.

Solution.--The extreme limits of motion of lever 4 relative to link 1 must, of course, be determined before the question can be settled. To find these limits we proceed as follows: (1) From 12 as a center with a radius equal to (12-23) + (23-34) strike an arc cutting at point 34_1 the circular path in which 34 moves relative to link 1. This, obviously, is one extreme position for the center of joint 34; (2) from 12 as a center with a radius (23-34) - (12-23) strike a second arc cutting the path of 34 at 34_2; this is the other extreme position of the center of joint 34. Moreover: (1) lines 14-34_1 and 14-34_2 are the two extreme positions of the center-line of lever 4; in other words, as crank 2 turns, clockwise or counterclockwise, through a complete turn the center-line of lever 4 will swing forward once and backward once through angle D_{41}. That is to say, while crank 2 turns through 360° lever 4 vibrates through the angle $2D_{41}$; moreover, evidently, when lever-pin center is at positions 34_1 and 34_2 crank-pin center is at positions 23_1 and 23_2, respectively.

Now, it ought to be evident, if the pin in question is represented by the smaller of the two circles shown at P_1 lever 4 as it swings through angle D_{41} will not foul it.

Problem 3.--Let the smaller of the two circles shown at P_3, Fig. 14, represent, for configuration C_2 a pin that is fastened to link 3 and projects out into and beyond the plane of motion of link 3 relative to link 4; determine whether or not link 4 will foul this pin.

Solution.--P_3' represents the pin in question when the mechanism is in its C_1 configuration; this is the case because triangles 23_1, 34_1, P_3' and 23_2, 34_2, P_3 are congruent. The figure makes it perfectly evident as soon as P_3' has been located that lever 4 will not foul it as this lever swings through angle D_{43} relative to link 3.

Problem 4.--For the lever-crank mechanism, Fig. 14, plot the locus L_1 described by point P_3 of link 3 relative to link 1 while crank 2 makes a complete turn, and then find the length of this path in inches.

Solution.--1. Assume a number of different positions of point 23 on the crank-pin circle, and for each of these positions find corresponding positions for points 34 and P_3, remembering that triangle 23, 34, P_3 remains unchanged in shape and size as the configuration of the mechanism changes. These several positions of point P_3 will be points on the required locus.

2. The length of the locus can be found with sufficient accuracy for the requirements of this problem by stepping off short chords on it and taking the sum of these chord lengths as the length of the path. At those portions of the path where the radius of curvature is comparatively small the chords must, the student should remember, be small, if the result is to be any good.

Problem 5.--Of a lever-crank 4-link linkage, Fig. 14, given: (1) The angle D_{41} through which lever 4 is to swing twice--once forward, once backward--for each rotation of crank 2; (2) the length of lever 4; (3) the ratio of the time of the slower to the time of the faster stroke of lever 4; (4) center-distance 14-12; determine: (1) The location of center 12; (2) the length of crank 2; (3) the length of connecting-rod 3.

Solution.--The figure shows the mechanism in its two extreme configurations; thus the angle, D_{41}, between the two extreme positions of lever 4 is, as has already been pointed out, half the angle through which lever 4 swings for each rotation of crank 2. Moreover, while crank 2 turns clockwise from position 23_1-12 to position 23_2-12, through angle D_{21}, lever 4 swings forward through angle D_{41}, and while crank 2 turns clockwise from position 23_2-12 to position 23_1-12, through angle $360-D_{21}$, lever 4 swings backward through angle D_{41}. Quite obviously also, the acute angle between 23_1-34_1 and 23_2-34_2 is

$$\psi = D_{21} - 180^\circ. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

Now let T_f represent the time of the forward stroke of lever 4 and T_r the time of its return stroke; then, if the crank turns with uniform angular velocity,

$$\frac{T_f}{T_r} = \frac{D_{21}}{360 - D_{21}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

that is,

$$D_{21} = \frac{360 T_f}{T_r + T_f}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

and from equation (11)--

$$\psi = \frac{360 T_f}{T_r + T_f} - 180^\circ. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

It is clear, too, from the figure that

$$(12\text{-}34_1) - (12\text{-}34_2) = 2(12\text{-}23_1),$$

that is,

$$12\text{-}23_1 = \frac{(12\text{-}34_1) - (12\text{-}34_2)}{2},$$

Thus we have the following construction:

Construction.--(1) Locate point 14 at any convenient position on the paper and through it draw lines 14-34_2 and 14-34_1 of the given length, making with each other the given angle D_{41}; (2) through points 34_2 and 34_1, draw a circle S_1 of such a radius that one half of arc

34_2-34_1 on this circle will measure angle ψ equation (14); (3) with 14 as a center and the given center-distance 14-12 as a radius strike an arc cutting circle S_1 at point 12; (4) join point 12 to 34_2 and 34_1; (5) from 12 as a center with 12-34_2 as a radius strike an arc cutting 12-34_1 at point G; then G-34_1 is twice the required length of the crank, and (12-34_2) plus the length of the crank is the length of the connecting rod.

Problem 6.--Consider the 6-link double slider-crank linkage, Fig. 15. Given: (1) the dimensions of the mechanism, (2) the position of the three coincident points P_1, P_4, and P_6, at the beginning of the period of displacement, that is, for the zero configuration, and (3) the displacement D_{21} of crank 2 relative to link 1; find: (1) D_{23}, (2) D_{25}, (3) D_{35}, (4) L_{41}, (5) L_{61}, (6) L_{46}.

Solution.--(1) Since lines 2_1 and 3_1 coincide in the zero configuration of the mechanism, the angle between these lines for configuration C_1, measured counterclockwise from 3_1, is D_{23}. (2) Draw line L_2 through point 25_1 making with line $2'_1$ the same acute angle that 2_0 makes with 5_0, then the angle between L_2 and 5_1, measured counterclockwise from line $5'$, is evidently D_{25}; because if line L_2 is fastened to link 2 it will evidently coincide with line 5_0 in the zero configuration. (3) Through point G, in which lines 3_1 and 5_1 meet, draw line L_3 so that the acute angle between it and 3_1 is equal to the acute angle between 5_0 and 3_0; then the acute angle between L_3 and 5_1, measured clockwise from 5_1, is D_{35}; because if line L_3 is fastened to link 3 it will be parallel to line 5_0 in the zero configuration, and thus will make the same angle with 3_0 that 5_0 does. (4) Coincident points P_1, P_4, and P_6 are supposed to be fastened to links 1, 4, 6, respectively but not, of course, to each other. Since, moreover, links 1 and 4 are paired to each other by a sliding pair, all points in link 4 have exactly the same displacement relative to link 1. The same statement is true for points in link 6. If, therefore, we draw $P_4P'_4$ and $P_6P'_6$ parallel and equal, respectively, to 34_0-34_1 and 56_0-56_1, the former two of these will be L_{41}, and L_{61}, respectively, and the line joining P'_6 to P'_4 will, according to Theorem V, be L_{46}.

Problem 7.--For the 6-link mechanism of the preceding problem plot the locus described by point P_4, Fig. 16, relative to link 6 for a complete rotation of crank 2 relative to link 6 for a complete rotation of crank 2 relative to link 1.

Solution.--For the configuration C_0 of the mechanism, the configuration, that is, for which crank-pin center 23 is at 23_0, let symbol $P_1P_4P_6$ designate three instantaneously coincident points in links 1, 4, and 6, respectively. Then when the configuration has changed to C_0 point P_4 has moved to ${}_1P_4$, relative to link 1, and point P_6 to ${}_1P'_6$ relative to link 1, and, point P_4 to point ${}_1P'_4$ relative to link 6. This we know from Theorem V. Similarly, for configurations C_2, C_3, C_4, C_5, etc. point P_4 will be at positions ${}_2P'_4$, ${}_3P'_4$, ${}_4P'_4$, ${}_5P'_4$, etc., respectively, relative to link 6. In other words, P_4 ${}_1P'_4$ ${}_2P'_4$ ${}_3P'_4$ ${}_4P'_4$ ${}_5P'_4$, etc., is the locus that point P_4 of link 4 describes in link 6 as the mechanism goes through a complete cycle.

In the figure crosshead 6 is expanded as shown and curve $P_4 \ldots {}_5P'_4$ is the center-line of a slot cut in link 6, thus expanded. Moreover, point P_4 is the center of a pin that slides in this slot.

The mechanism shown in Fig. 17, is the well-known and much-used 3-link spur-gear train of two gears and one baselink. We call it the external cubic spur-gear mechanism. The gears are paired to each other by *higher* pairing and to the baselink by *lower* pairing. Thus the mechanism contains three pairs: one higher and two lower. The circles shown in the figure, tangent to each other at P_2P_3, are called pitch circles. These circles play a very important role in spur-gear calculations but they do not appear on the finished gears. The teeth, which slide on each other, are supposed to be so shaped and so placed in relation to the pitch circles that they cause these latter to roll on each other with perfect rolling contact. Let us solve the following problem of displacement concerning these gears.

Problem 8.--For the 3-link spur-gear mechanism, Fig. 17, given the angle through which gear 2 turns relative to baselink 1 in time t, find (1) the angle, measured in revolutions, through which gear 3 turns relative to baselink 1 in this time, and (2) the angle through which gear 3 turns relative to gear 2 in the same time.

Solution.--Let gear 2 turn as shown through the angle D_{21}. Then because the pitch circles *roll* on each other without slip, make arc P_3Q_3 equal to arc P_2Q_2, and thus determine the angle, D_{31}, through which gear 3 turns relative to baselink 1 while gear 2 turns through the given angle D_{21}. Now by Theorem IV as well as from triangle 12, 13, S we have

$$D_{23} = D_{13} + D_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

Also,

$$arc\ P_2Q_2 = arc\ P_3Q_3, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

that is,

$$R_2D_{21} = R_3D_{13}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

therefore

$$D_{23} = \frac{R_2}{R_3}D_{21} + D_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

But, evidently,

$$\frac{R_2}{R_3} = \frac{2\pi R_2}{2\pi R_3} = \frac{\frac{2\pi R_2}{P'_{23}}}{\frac{2\pi R_3}{P'_{23}}} = \frac{Z_2}{Z_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

in which P'_{23} is the circular pitch of the gears, that is, the distance, as shown in the figure, measured along the pitch circle between corresponding points on adjacent teeth, and Z_2 and Z_2 the numbers of teeth on the two gears. Thus, equations (18) and (19),

$$D_{23} = \frac{Z_2}{Z_3}D_{21} + D_{21}$$

$$= D_{21} \times \frac{Z_2 + Z_3}{Z_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

a formula that gives D_{23}, the angular displacement of gear 2 relative to gear 3, in terms of the angular displacement of gear 2 relative to link 1 and the numbers of teeth on the gears.

Comments.--1. Obviously it is immaterial to equation (20) whether D_{23} and D_{21} are measured in radians, degrees, or revolutions.

2. In deriving equation (20) it was tacitly assumed that link 1 was held stationary while gears 2 and 3 turned relative to it through the angles D_{21} and D_{31}. But of course it doesn't make any difference as far as the relation between D_{23} and D_{21} is concerned which of the three links is supposed to be the fixed one, provided only that the given displacement D_{21} is the same in magnitude and sense in all three cases. Let us assume in illustration of this fact that gear 2, Fig. 18, is the fixed link.

Since when link 1 was fixed we assumed gear 2 to turn counterclockwise relative to link 1 through angle D_{21}, we must now, when gear 2 is fixed, assume link 1 to turn clockwise relative to gear 2 through angle D_{12}. To find the angle through which gear 3 turns relative to link 1 while link 1 turns through the angle D_{12} relative to gear 2, we must, of course, just as before, make

$$arc\ P_1Q_1 = arc\ Q_1Q_3. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

Then from Theorem V, as well as from triangle 12, 13_2, S, we have

$$D_{32} = D_{12} + D_{31}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

or, what is the same thing,

$$D_{23} = D_{13} + D_{21}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

from which, proceeding just as before, we get equation (20).

3. When gear 2 is fixed and base-link 1 turns around axis 12, gear 3 rolls around on the outside of gear 2, without slip if the teeth of the gears are correctly shaped and spaced, and we have then what is called an epicyclic or planetary gear train, a kind of train that belongs to a class that is much used in practice.

4. Evidently equation (20) may be written

$$\frac{D_{23}}{D_{21}} = \frac{D_{32}}{D_{12}} = \frac{Z_2 + Z_3}{Z_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

a form of equation (20) which tells us that when gear 2 is held fast and baselink 1 is turned about axis 12 through the angle D_{12} (of *any size whatever*, it should be remembered) gear 3 turns in the same time through the angle.

$$\frac{Z_2 + Z_3}{Z_3} D_{12}.$$

Problem 9.--For the internal cubic spur-gear mechanism, Fig. 19, derive the relation that corresponds to equation (20) for the preceding problem.

Solution.--In Fig. 19 make

$$arc\ P_3Q_3 = arc\ P_2Q_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

and thus determine displacement D_{31}. Then from triangle 12, 13, S, in accordance with Theorem IV, we have

$$D_{31} = D_{21} + D_{32}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

for which, reasoning just as we did in deriving equation (20), there results the relation

$$D_{32} = \frac{Z_2 - Z_3}{Z_3} D_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

Comments.--1. Formula (27) differs from formula (20) only very slightly, let the student notice. The one involves the difference of Z_2 and Z_3, the other the sum.

2. The student should be sure to understand and appreciate the kinematic meaning of these two formulas.

3. Equation (27) may be written in the form

$$\frac{D_{32}}{D_{21}} = -\frac{D_{32}}{D_{12}} = \frac{Z_2 - Z_3}{Z_3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (27')$$

a form which shows that when baselink 1 turns relative to gear 2 through the angle D_{12} in one sense gear 3 turns relative to gear 2 through the angle

$$\frac{Z_2 - Z_3}{Z_3} D_{12}$$

in the opposite sense.

B. Complanar Velocity

There are two kinds of complanar velocity with which we have to deal in the science of mechanism: (1) *complanar linear velocity* and (2) *complanar angular velocity*. To the consideration of these two sorts of velocity, both of which are of the utmost importance to the student of mechanism, we now turn.

1. *Complanar Linear Velocity.*--When a point moves relative to a rigid body of reference it describes a path that belongs to the body of reference. For the sake of shortness we often say that it describes a path *in* the body of reference. The tangent to the path at the position where the moving point is at any instant gives the direction of the velocity of the point at the instant. The time rate with which at any instant the point is describing its path is the magnitude of the instantaneous linear velocity of the point relative to the body in which lies the path along which the point is moving.

A linear velocity, the student must be sure to understand, possesses four characteristics and every single one of them has to be taken into account if we are to deal with the velocity intelligently and effectively. These characteristics are: *magnitude, direction, sense,* and *instantaneousness*. Linear velocity is thus a *vector* quantity, not a *scalar* quantity.

It is, moreover, of the greatest importance for the student of mechanism to realize that there are two other things that he must know about a linear velocity in order to solve with facility problems in which it is involved. They are: (a) the body that carries the point whose linear velocity is under consideration, (b) the body of reference, the body, that is, to which belongs the path along which the point is moving at the instant under consideration.

Let us illustrate this idea of complanar linear velocity with a diagram, Fig. 20. In this figure let α_1 be the path in body 1 along which point P_2 of body 2 moves, let O, on the path, be the origin from which distances moved by point P_2 are measured, and let L_1 be the distance the point moves in time T_1 and L_2 the distance it moves in time T_2. Let P_2p and $P_2'p'$, tangents to the path at P_2 and P_2', respectively, represent the velocity of the point at these two positions. Choose a pole S and through it draw St and St' parallel and equal to P_2p and $P_2'p'$, respectively. Then vector tt' represents the change in the velocity of point P_2 during time T_2-T_1, because this is the velocity that has to be added vectorially to St in order to give St'. We say that St' is the vector sum of St and tt', meaning that these three directed quantities when laid off in the proper directions and to the same scale form a triangle. Now make Sr equal to St; then rt', the scalar difference between St' and St, is the change in the *magnitude* of the velocity of point P_2 in time T_2-T_1; in other words, rt' represents the change in the *speed* of point P_2 that occurs while this point moves along its path from position P_2 to position P_2'.

It is vitally important to distinguish carefully between the meanings of these two quantities, tt' and rt'. The first is the total change in the *velocity* of point P_2 during time T_2-T_1, the other is the total change in the *speed* of this point in this time. The former is a vector quantity, the latter a scalar quantity.

To symbolize the velocity of point P_2 at a given instant we use the symbol ${}_{P_2}V_{21}$; to symbolize the average speed of a point P_2 during a finite interval of time we use the symbol ${}_{P_2}\overline{S}_{21}$.

If L_1, Fig. 20, is the distance in feet that point P_2 moves along its path α_1, in T_1 minutes and L_2 the distance it moves in T_2 minutes, then the average *speed* in feet per minute of the point while it is moving from position P_2 to position P_2', or, what is the same thing, its average speed during the time interval T_2-T_1, is evidently

$$ {}_{P_2}\overline{S}_{21} = \frac{L_2-L_1}{T_2-T_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (28) $$

The average speed of the point during time T_2-T_2 is thus the constant speed with which the point would have to move from the beginning to the end of the time interval in order to cover the distance L_2-L_1 in the time that it actually does take to cover it. The point may not move during the time interval in question for any finite portion of time with this average speed, but if it was moving with this speed when in position P_2 and kept to this speed in going from P_2 to P_2', it would cover the distance L_2-L_1 in time T_2-T_1.

Now if we imagine the distance-interval L_2-L_1 to become smaller and smaller, the corresponding time-interval T_2-T_1 will become smaller and smaller; and, finally, if we imagine the distance-interval L_2-L_1 to become infinitely small, the corresponding time-interval will also become infinitely small, and then in the notation of the differential calculus we shall have

$$P_2\bar{S}_{21} = \frac{ds}{dt}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

that is to say, we shall have the average speed with which point P_2 moves over the infinitely short distance L_2-L_1(=ds) in the infinitely short time T_2-T_1(=dt). But the point moves over this infinitely short distance with constant speed; moreover, the direction in which the point moves during this infinitesmal portion of time remains unchanged; hence we call its average speed in this case its instantaneous linear velocity, and for P_2S_{21} in equation (29) we substitute the symbol P_2V_{21}, and write

$$P_2V_{21} = \frac{ds}{dt}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

Comment.--Let P_1, Fig. 20, be the point of body 1 with which the moving point P_2 of body 2 coincides at the instant when P_2 is at position P_1 on its path; then evidently the instantaneous linear velocity of point P_2 relative to body 1 is equal and opposite to the instantaneous linear velocity of point P_1 of body 1 relative to body 2; or, symbolically

$$P_2V_{21} = -P_1V_{12}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

Equation (31), the student should be sure to observe, does not tell us which of the two quantities in it is positive or which is negative; it says only that the quantities are of opposite sign.

But, on the other hand, the equation does show us that if we alter the order of the numerals in the subscript of P_2V_{21} we must not only change the sign in front of the symbol but also the left-hand subscript of the symbol.

2. *Complanar Angular Velocity.*--Consider the complanar motion of the two rigid bodies shown in Fig. 21. On each of these bodies, which may be of any size and any shape, is formed a plane surface, and anywhere on each of these plane surfaces is scribed a straight line: L_1 on body 1, L_2 on body 2, so that, supposing the two plane surfaces to be in plane contact with each other, these two lines lie in the same plane and thus intersect each other at some angle, θ, say. When these bodies undergo complanar relative motion in any manner whatever the angle θ between these two lines changes, and the time-rate at which it is changing at any instant is the instantaneous angular velocity of the two bodies.

An angular velocity possesses four characteristics. They are: (a) magnitude, (b) direction, (c) sense, (d) instantaneousness. If the motion is complanar, as we assume it to be at present, the second characteristic hardly needs attention, because it is fixed and the same for all the bodies of the system; but each of the other three characteristics needs careful consideration in every problem in which an angular velocity occurs.

Of course it will be understood that no conception of an angular velocity can be formed with only one rigid body under consideration. In other words, the very term, angular velocity, implies the presence of at least two bodies: the moving body and the body of reference.

Returning to Fig. 21, let θ_1 be the value of angle θ at time T_1 and θ_2 its value at subsequent time T_2; then for the average angular velocity of body 2 relative to body 1 during time T_2-T_1 we have the expression

$$\frac{\theta_2-\theta_1}{T_2-T_1} = \bar{\omega}_{21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

in which the θ's are measured in radians, the times in minutes, and $\bar{\omega}_{21}$, the average angular velocity, in radians per minute. Of course we may write

$$\frac{\theta_2-\theta_1}{T_2-T_1} = 2\pi\bar{N}_{21} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

where N_{21} is the average angular velocity of body 2 relative to body 1, in revolutions per minute, because

$$\bar{\omega}_{21} = 2\pi\bar{N}_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (34)$$

Equation (34) is true because in each revolution there are 2π radians.

If we now assume the change in θ to become smaller and smaller, the difference in the two corresponding times will simultaneously become smaller and smaller, and if, finally, we assume $\theta_2-\theta_1$ to become infinitely small, the corresponding time T_2-T_1 will become infinitely small, and then in the symbolism of the differential calculus we have

$$\frac{d\theta}{dt} = \omega_{21}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

that is to say, we have the expression for the average angular velocity of body 2 relative to body 1 in the infinitely short time dt. But during this infinitesmal portion of time the magnitude of the angular velocity of body 2 remains constant, and thus we call it the *instantaneous*, not average, angular velocity of body 2 relative to body 1, and write

$$\frac{d\theta}{dt} = \omega_{21} = 2\pi N_{21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (36)$$

in which ω_{21}, freed of the overscore, is the *instantaneous* angular velocity of body 2 relative to body 1 in radians per minute, and N_{21} the angular velocity in revolutions per minute.

Comments.--1. The student must not think that bodies 1 and 2, Fig. 21, rotate relative to each other about the axis passing through point O in which lines L_1 and L_2 intersect each other. This cannot, in general, be the case because lines L_1 and L_2 are chosen perfectly arbitrarily in the planes formed on bodies 1 and 2.

2. This concept of complanar angular velocity is a matter of great importance to the student of mechanism because the links of most machines move relative to each other with complanar relative motion. Almost all of the links in a steam engine or gas engine, for example, move relative to each other in this manner.

3. If the successive changes in angle θ, Fig. 21, in successive equal intervals of time are equal, the angular velocity of body 2 relative to body 1 is constant, and then, of course,

$$\bar{\omega}_{21} = \omega_{21} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (37)$$

and

$$\bar{N}_{21} = N_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (38)$$

In solving certain problems of mechanism we often make the assumption that some angular velocity involved in the problem is constant, but the student must remember, in such cases, that we are then making an assumption that cannot be exactly realized in practice; because it is utterly beyond our engineering powers to make one link of a mechanism move with constant angular velocity relative to some other link. As a matter of fact, a constant angular velocity does not occur anywhere in nature. Even the earth does not rotate on its axis relative to the sun with exactly uniform angular velocity, so the astronomers tell us.

4. If at a given instant body 2 rotates relative to body 1 at the rate of N_{21} rpm, clockwise, then evidently body 1, at the same instant, rotates relative to body 2 at the same rate but in the opposite sense. That is to say symbolically,

$$N_{21} = -N_{12}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (39)$$

Thus we see that if we change the order of the numerals in the subscript of the symbol N_{21} we must also change the sign of the symbol.

The student must not make the mistake of thinking that equation (39) fixes the sign of either N_{21} or N_{12}. The equation asserts only that the signs of these two quantities are opposite; it says nothing about which one is positive or which one is negative.

3. *The Relation Between Complanar Linear Velocity and Complanar Angular Velocity.*--To establish this relation we need the following important theorem:

Theorem VI.--All complanar motion of one rigid body relative to a second rigid body is rotation.

Proof.--1. When one rigid body undergoes constrained relative motion relative to another rigid body the points of each body describe plane paths--straight or curved--*in* the other body. And, in general, if the paths described by two points of the first body relative to the second body are known, the relative displacement of the two bodies is thereby determined.

To show this consider Fig. 22, in which α_1 and β_1 are the complanar paths described simultaneously in body 1 by points P_2 and Q_2 of body 2. Then we have to show that in general for each point on path α_1 only one point on path β_1 corresponds. Let 2_1 $(=P_2Q_2)$ be any position of body 2, subject only to the conditions that P_2 shall lie on path α_1 and point Q_2 on path β_1. Choose a second point P'_2 on path α_1 at a small but finite distance from P_2, and about this point as center with a radius equal to the constant length P_2Q_2 strike the arc Cq, cutting path β_1 at Q'_2. Then the position of point Q'_2 and thus also the corresponding position 2_2 of body 2 will in general be uniquely determined; because although circular arc Cq may cut path β_1 in other points besides point Q'_2, the conditions of the relative displacement of the two bodies will generally exclude such other points. To prove the truth of this latter statement consider point R_2 a second point in which circular arc Cq cuts path β_1. Obviously this point must be rejected, because the only points on path β_1 between points Q'_2 and R_2 that correspond to points on path α_1 between P_2 and P'_2 are those points on path β_1 that lie between Q'_2 and Q_2. The student can easily satisfy himself of the truth of this statement by striking an arc with radius P_2Q_2 from any point on path β_1 between points Q_2 and R_2. He will then see that this arc does not cut path α_1 between P_2 and Q_2.

Now let α_1 and β_1, Fig. 23, be the complanar paths belonging to body 1 along which points P_2 and Q_2 of body 2 move simultaneously as the two bodies undergo complanar relative motion. Let 2_1 and 2_7 represent any two possible positions of body 2. Theorem I, the student will remember, has shown us that for complanar displacement or motion a straight line can represent a rigid body. Moreover, Theorem II has shown us that body 2 can be moved from the first to the second of these two positions by a single rotation about an axis passing through 12' perpendicular to the plane of motion. When so moving, however, points P_2

and Q_2 of body 2 do not and cannot travel along the paths α_1 and β_1, respectively; they travel, instead, along concentric circular arcs, not shown, of which 12' is the center. But, it is evident, the closer the two positions of body 2 are to each other the more nearly these circular arcs will come to coinciding with the paths α_1 and β_1. Take in illustration of this statement positions 2_1 and 2_3, which are much closer to each other than positions 2_1 and 2_7. Drawing, just as for Fig. 3, Theorem II, the perpendicular bisectors of the chords $P_2'P_2$ and $Q_2'Q_2$ we get in point *12"*, the point of intersection of the perpendicular bisectors, the point about which body 2 must rotate to move from position 2_1 and 2_3 by a single rotation. For these two much closer positions of body 2 the two concentric circular arcs, $P_2'P_2$ and $Q_2'Q_2$, swept from *12"* as a center, along which points P_2 and Q_2 travel when body 2 rotates in going from position 2_1 to position 2_3 by a single rotation, *practically* coincide with paths α_1 and β_1.

Now let us pass to the limit and assume that positions 2_1 and 2_3 of body 2 are infinitely close positions. Then, evidently, the concentric circular arcs, $P_2'P_2$ and $Q_2'Q_2$, swept from the point of intersection, 12, of the perpendicular bisectors of the chords $P_2'P_2$ and $Q_2'Q_2$ will *exactly* coincide with paths α_1 and β_1 over the infinitely short arcs $Q_2'Q_2$ and $P_2'P_2$; that is to say, these concentric circular arcs, whose center is 12, will be tangent to paths α_1 and β_1 at points Q_2 and P_2, respectively, and the perpendicular bisectors of the infinitely short arcs $Q_2'Q_2$ and $P_2'P_2$ will be the normals to these paths at points Q_2 and P_2, respectively.

Thus we see that for body 2 to move from position 2_1 to the infinitely close position 2_3 with a single rotation it must rotate about the axis that passes through the point of intersection of the normals to the paths α_1 and β_1 at points P_2 and Q_2, respectively. And since this condition of things is true wherever points P_2 and Q_2 may be on their paths, α_1 and β_1, we conclude that the motion of body 2 relative to body 1 is a continuous series of instantaneous infinitesmal rotations about successive, different axes, such as axis 12, for example. Moreover, since paths α_1 and β_1 are perfectly general paths, we are led to the conclusion that the theorem as stated is true.

Comments on Theorem VI.--1. A little further consideration of Fig. 23 ought, we think, to make it perfectly clear to the student that to find the velocity-center, and thus also the velocity-axis, of two bodies that are undergoing at a given instant instantaneous complanar motion relative to each other we need know only the direction of motion of each of two points in one of the bodies relative to the other. Suppose, in illustration of this statement, that we consider body 2, Fig. 23, in its position 2_1. At the instant, points P_2 and Q_2 are moving along paths α_1 and β_1, respectively, and velocity-center 12 is thus, as we have seen, the point of intersection of the normals to these paths at P_2 and Q_2. But these normals are perpendicular, respectively, to the tangents, TT and $T'T'$, to the paths at P_2 and Q_2, and these tangents are, of course, the directions in which points P_2 and Q_2 are moving relative to body 1 at the instant. Thus the truth of the statement becomes evident.

2. If, to imagine a case that often arises in practice, paths α_1 and β_1 are concentric circular arcs, the normals to them at points P_2 and Q_2 will intersect in the same fixed point wherever points P_2 and Q_2 may be on their paths; that is to say, body 2 then rotates relative to body 1 about a permanent axis.

Definition.--The axis about which body 2 rotates relative to body 1 at any instant is called a velocity-axis: an instantaneous velocity-axis, if rotation takes place about it only for an instant; a permanent velocity-axis if rotation about it lasts for a finite period of time.

Definition.--The point in which a velocity-axis pierces the plane of motion is called a velocity-center. It is instantaneous or permanent according as the corresponding velocity-axis is instantaneous or permanent.

If the two moving bodies are numbered 1 and 2 the velocity-axis is designated with the symbol 12 or 21. Similarly, if the two bodies under consideration are numbered with any other numerals.

3. It is quite unscientific, but often very convenient, to speak of a rigid body as rotating about a velocity-center. Such language is unscientific because no such thing is possible. When a rigid body seems to be rotating about a point it is really rotating about a straight-line axis through the point, not about the point. Hereafter, therefore, when we speak of a rigid body as rotating about a velocity-center we really mean that the body is rotating about the velocity-axis through that point.

A velocity-center is designated with the same symbol as that used to designate the velocity-axis.

4. We saw when considering Theorem II that the axis about which body 2 rotates in going from one to any other position remained fixed in both bodies during the rotation. In exactly the same way and for exactly the same reasons, the axis about which two bodies rotate for an instant only remains fixed in the two bodies for the instant. Thus it belongs to the two bodies for the instant; and therefore

Property A of the Velocity-Axis.--The linear velocity of any point on the velocity axis of two bodies that are undergoing instantaneous complanar motion relative to any third body is the same relative to this third body in whichever one of the first two bodies the point is considered to be.

This property of the velocity-axis is perhaps the most fruitful idea in the whole of elementary kinematics. The student is therefore strongly urged to ponder it until the truth of it is as evident to him as is the proposition that twice two is four.

When a rigid body 2 undergoes instantaneous complanar motion relative to a rigid body 1 the points in body 2 move along their respective paths in body 1, and at any instant body 2 has a certain definite angular velocity relative to body 1, and at the same instant each point in body 2 has a certain definite linear velocity relative to body 1. It is our purpose now to find the relation between these two velocities. We shall designate the relation *Property B of the Velocity-Axis.*

Property B of the Velocity-Axis.--Let body 2, Fig. 24, be rotating relative to body 1, at the instant under consideration, about the instantaneous axis 12 at the rate of N_{21} revolutions per minute. Let P_2 be any point in body 2, and let P_2V_{21}, in feet per minute, represent the magnitude of the linear velocity of point P_2 relative to body 1. Then if we measure (P_2-12) in feet, the relation in question is

$$P_2V_{21} = 2\pi N_{21}(P_2-12); \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

because if at the instant body 2 is rotating relative to body 1 about axis 12 at the rate of N_{21} rpm, and if all the conditions remained constant for one minute, point P_2 would move through the distance $2\pi N_{21}(P_2-12)$ in that time; that is to say, in one minute P_2 would move over N_{21} circumferences the length of each one of which would be $2\pi(P_2-12)$. But the distance that a point moves in one minute when its speed is constant for the minute is evidently the magnitude of the velocity of the point at any time during the minute.

Caution.--The student must be on his guard against thinking that body 2 does actually continue to turn for one minute around axis 12 at the constant rate of N_{21} rpm. That will actually be the case only when axis 12 is a permanent axis and N_{21} is constant for a finite period of time.

Of course we may write equation (40) as follows:

$$P_2V_{21} = \omega_{21}(P_2-12), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (41)$$

since as we have already seen,

$$\omega_{21} = 2\pi N_{21}.$$

If we take another point, Q_2, anywhere in body 2, we have for it likewise

$$Q_2V_{21} = 2\pi N_{21}(Q_2 - 12); \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (42)$$

thus

$$\frac{P_2V_{21}}{Q_2V_{21}} = \frac{P_2 - 12}{Q_2 - 12}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (43)$$

that is to say, the linear velocities of two points in the same rigid body are directly proportional to their distances from the axis of rotation, a principle of which the student will have to make constant use in this course.

Evidently the relations just proved are true whether axis 12 is instantaneous or permanent.

4. *Angular Velocity Relations.*--So far in what has been said about velocity, whether linear or angular, only two rigid bodies have been involved in the relations derived. Now we are to study some very important theorems, of the greatest use in solving problems in mechanism, involving three rigid bodies.

The first of these theorems I call the *Kennedy-Aronhold* theorem of three velocity-centers for three rigid bodies undergoing instantaneous complanar relative motion, because it was discovered simultaneously and independently by Kennedy in England and Aronhold in Germany, about sixty years ago. It is of great use--in fact indispensable--in finding velocity-centers, and these latter play a most important role in nearly all velocity and acceleration determinations.

Theorem VII.--The three velocity-centers of three rigid bodies undergoing instantaneous complanar relative motion are collinear.

Proof.--There must be *three* centers because according to Theorem VI there is one center for each pair of bodies and there are three pairs of bodies, evidently. Let the bodies, numbered 1,2,3, be as shown in Fig. 25a; and at first assume the three centers not in the same straight line.

Since, Theorem VI, Property A, center 23 is a point in body 2 at and for the instant, and since, moreover, body 2 is rotating relative to body 1 around axis 12, center 23 must be moving relative to body 1 in the direction given by line $23 - t_2$, drawn through point 23 at right-angles to line 12 - 23. But, Theorem VI, Property A, point 23 is also a point in body 3 at and for the instant. And as a point in body 3 it must be moving relative to body 1 in the direction given by line $23 - t_3$, drawn through point 23 perpendicular to line 23 - 13. But a point (23 in this case) cannot move in two different directions at the same instant relative to the same body of reference (body 1 in this case). Therefore point 23 must be so situated relative to centers 12 and 13 that directions $23 - t_2$ and $23 - t_3$ can coincide; and the only position for point 23 for which lines $23 - t_2$ and $23 - t_3$ can coincide is evidently some position on line 12-13. Thus the theorem is true.

The three centers are shown correctly positioned in Fig. 25b.

Comments.--1. The theorem doesn't tell us, the student should be sure to observe, where on line 12-13 center 23 must lie; it tells us only that it must be *somewhere* on this line. The next theorem, Theorem VIII, will determine the relative positions of centers 12, 23, and 13 on line 12-13.

2. Since the three corresponding velocity-axes are all perpendicular to the plane of motion they are parallel, and since, Theorem VII, the three points in which they pierce the plane of motion are collinear, it follows that the three velocity-axes are complanar.

Before considering the next theorem we must, if we are to avoid prolixity of expression and the risk of being obscrure, define a few terms that we shall have to use in stating and proving the theorem.

Angular Velocity Ratios.--Two kinds of angular velocity ratios occur in the analysis of mechanisms: cubic and quartic, I call them. They are defined as follows:

Definition VIII: *Cubic Angular Velocity Ratio.*--An angular velocity ratio involving three, but not more nor fewer than three, links of the mechanism under consideration is called a cubic angular velocity ratio.

Examples:

$$\frac{N_{21}}{N_{31}}, \quad \frac{N_{45}}{N_{65}}, \quad \frac{N_{79}}{N_{89}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (A)$$

Comments.--1. Each of these ratios involves, the symbolism shows, only three links, and one link in each ratio is common to the two angular velocities that compose it.

2. The link that is common to the two angular velocities of the ratio is called the link or body of reference; the other two links of the ratio are called the moving links or bodies.

But the student must not think that the link or body of reference must be "fixed" in order to be a reference body. In many cases it is immaterial whether we think of it as fixed or not, but in some cases such a restriction would not, as we shall see, be allowable.

3. The subscript of the numerator of the ratio always designates the velocity-axis about which the two bodies designated by the numerals of the subscript rotate relative to each other, either instantaneously or permanently. Similarly for the subscript of the denominator of the ratio.

4. The velocity-center designated by the subscript of the numerator of the ratio is called the *primary* center; the velocity-center designated by the subscript of the denominator of the ratio is called the *secondary* center; and the velocity-center designated by the symbol that results from combining the first numeral of each subscript is called the *mutual* center.

Example: The symbols 21, 31, and 23 of the ratio N_{21}/N_{31} designate, respectively, the primary, the secondary, and the mutual center for the three rigid bodies 1, 2, 3 with which the ratio is associated.

The order in which the numerals are placed in a symbol designating a velocity-center is immaterial: for instance, for the three centers in question we may write either 21, 31, 23, or 12, 13, 32.

5. *Convention of Signs.*--(a) The sign of a cubic angular velocity ratio will be positive when the two moving bodies of the ratio turn in the same sense relative to the body or link of reference.

(b) The sign of a cubic angular velocity ratio will be negative when the moving bodies of the ratio turn in opposite senses relative to the body or link of reference.

The conditions of the problem in any case will always tell us whether the two moving bodies or links of the mechanism are turning in the same sense or in opposite senses relative to the body or link of reference; thus it will always be possible to fix the sign of a cubic ratio.

Definition IX: *Quartic Angular Velocity Ratio.*--An angular velocity ratio involving four links of the mechanism under consideration is called a quartic angular velocity ratio.

Examples:

$$\frac{N_{21}}{N_{34}}, \quad \frac{N_{45}}{N_{67}}, \quad \frac{N_{87}}{N_{89}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (B)$$

Comments.--1. Each of these ratios involves, the symbolism shows, four links or bodies; the symbolism shows, moreover, that no link in any one of the ratios is common to the two angular velocities that compose it. In other words, a quartic ratio has no single or common link of reference, but two different links of reference. Links 1 and 4 in the ratio N_{21}/N_{34}, for example, are both links of reference: link 1 for the angular velocity N_{21} and link 4 for the angular velocity N_{34}.

Just as for the cubic angular velocity ratio, the subscript of the numerator of a quartic angular velocity designates the velocity-axis (instantaneous or permanent, as the case may be) about which the two bodies or links designated by the numerals of the subscript rotate relative to each other. And similarly for the subscript of the denominator.

The subscript 21 or 12, for example, in the quartic ratio N_{21}/N_{34} designates the velocity-axis about which bodies or links 1 and 2 rotate at the instant relative to each other. Similarly, the subscript 34 or 43 designates the velocity-axis about which bodies or links 3 and 4 rotate at the same instant relative to each other.

3. The *second* numeral in each subscript of a quartic angular velocity ratio symbol designates the body of reference for the angular velocity to which it belongs. The *first* numeral in each subscript designates the moving body for the corresponding angular velocity.

4. *Rule of Signs for Quartic Ratios.*--The sign of a quartic angular velocity ratio is positive when the two moving bodies both turn in the same sense relative to their bodies of reference. If, for instance, for the quartic ratio N_{21}/N_{34} links 2 and 3 both turn in the same sense relative to reference links 1 and 4, respectively, the sign of the ratio is positive. If, on the other hand, the moving links turn in opposite senses relative to their respective links of reference, the sign of the ratio is negative. N_{21}/N_{34}, for example, is negative if links 2 and 3 turn in opposite senses relative to reference links 1 and 4, respectively.

Theorem VIII: *The Theorem of Three Cubic Angular Velocity Ratios of Three Rigid Bodies Undergoing Complanar Relative Motion.*--The ratio of the angular velocities of any two of three rigid bodies undergoing simultaneous complanar motion relative to a third is equal to the inverse ratio of the two linear segments into which the mutual center divides the distance between the other two centers. That is to say, symbolically

$$\frac{N_{21}}{N_{31}} = \frac{23 - 31}{23 - 21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (44)$$

$$\frac{N_{12}}{N_{32}} = \frac{13 - 32}{13 - 12}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (45)$$

$$\frac{N_{13}}{N_{23}} = \frac{12 - 23}{12 - 13}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (46)$$

where the positive or the negative sign is to be chosen in any one of the three cases according as the moving bodies turn in the same sense or in opposite senses relative to the body of reference.

Proof.--A. Let the three rigid bodies or links in question be represented by the contours shown in Fig. 25b. Moreover, let bodies 1 and 2 be rotating at the instant relative to each other about velocity-axis 12 at the rate of N_{21} rpm.; let bodies 2 and 3, likewise, be rotating at the same instant relative to each other about velocity-axis 23 at the

rate of N_{23} rpm.; and similarly, bodies 1 and 3 about velocity-axis 13 at the rate of N_{31} rpm.

1. Now, first, regard body 1 as the body of reference and bodies 2 and 3 as the moving bodies. Then since center 23 is a point common to bodies 2 and 3 for the instant, we have, Theorem VI, Properties A and B,

$$_{23}V_{21} = 2\pi N_{21}(23 - 21), \quad \ldots\ldots\ldots\ldots \quad (47)$$

$$_{23}V_{31} = 2\pi N_{31}(23 - 31). \quad \ldots\ldots\ldots\ldots \quad (48)$$

But a point (23 in this case) cannot move with two different linear velocities at the same instant *relative to the same body of reference* (body 1 in this case), thus

$$_{23}V_{21} = {}_{23}V_{31}. \quad \ldots\ldots\ldots\ldots \quad (49)$$

and therefore also, of course,

$$2\pi N_{21}(23 - 21) = 2\pi N_{31}(23 - 31), \quad \ldots\ldots\ldots \quad (50)$$

or, putting the preceding result in the form of a proportion,

$$\frac{N_{21}}{N_{31}} = \frac{23 - 31}{23 - 21}, \quad \ldots\ldots\ldots\ldots \quad (51)$$

which, neglecting for the moment the question of signs, is the same as equation (44).

2. Secondly, regarding body 2 as the body of reference and bodies 1 and 3 the moving bodies, and appealing again to Properties A and B of Theorem VI, we have

$$_{13}V_{12} = 2\pi N_{12}(13 - 12), \quad \ldots\ldots\ldots\ldots \quad (52)$$

$$_{13}V_{32} = 2\pi N_{32}(13 - 32); \quad \ldots\ldots\ldots\ldots \quad (53)$$

and for exactly the same reasons as those given in the preceding case,

$$_{13}V_{12} = {}_{13}V_{32}; \quad \ldots\ldots\ldots\ldots \quad (54)$$

thus

$$2\pi N_{12}(13 - 12) = 2\pi N_{32}(13 - 32), \quad \ldots\ldots\ldots \quad (55)$$

that is,

$$\frac{N_{12}}{N_{32}} = \frac{13 - 32}{13 - 12}, \quad \ldots\ldots\ldots\ldots \quad (56)$$

3. Thirdly, regarding body 3 as the body of reference and bodies 1 and 2 the moving bodies, the five equations of the proof, neglecting connectives, are

$$_{12}V_{13} = 2\pi N_{13}(12 - 13), \quad \ldots\ldots\ldots\ldots \quad (57)$$

$$_{12}V_{23} = 2\pi N_{23}(12 - 23), \quad \ldots\ldots\ldots\ldots \quad (58)$$

$$_{12}V_{13} = {}_{12}V_{23}, \quad \ldots\ldots\ldots\ldots \quad (59)$$

$$2\pi N_{13}(12 - 13) = 2\pi N_{23}(12 - 23), \quad \ldots\ldots\ldots \quad (60)$$

$$\frac{N_{13}}{N_{23}} = \frac{12 - 23}{12 - 13}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (61)$$

the last of which is the same as equation (46).

B. *Convention of Signs for Cubic Angular Velocity Ratios.*--We must now attend to the question of signs for equations (51), (56), and (61), because without signs these equations would be of little service to us in solving practical problems in mechanism.

To introduce signs into equations (51), (56), and (61) we must, just as in any other algebra, have a convention of signs. We adopt the convention already referred to, page 24, which we restate as follows:

Convention of Signs.--(1) If the two moving bodies or links turn at the instant in the same sense relative to the link or body of reference, the sign of the ratio is positive.

(2) If the two moving bodies or links turn at the instant in opposite senses relative to the link or body of reference, the sign of the ratio is negative.

1. Now consider Fig. 25b again, and let bodies 1 and 2 be moving bodies and body 3 the body of reference. Let us suppose, moreover, that body 1 is turning *clockwise* relative to body 3 about axis 13. Then the vector 12-a, drawn through 12 perpendicular to 12 - 13, and pointing as shown, represents the linear velocity of point 12, a point in body 1, relative to body 3. But, Theorem VI, Property A, point 12, the mutual center, being the velocity-center of bodies 1 and 2, is a point for the instant not only in body 1 but also in body 2; therefore vector 12-a represents the velocity, relative to body 3, of point 12 as a point in body 2. But if vector 12-a represents the velocity of 12 as a point in body 2, body 2 must be turning *clockwise* about axis 23 relative to body 3; that is to say, bodies 1 and 2 are turning in the same sense relative to body 3. We see, moreover, that as long as mutual center 12, the center that the two moving bodies, 1 and 2, have in common, lies anywhere except between centers 23 and 13 bodies 1 and 2 will both turn in the same sense relative to body 3.

2. In the next place, let bodies 2 and 3 be the moving bodies and body 1 the body of reference. Let us suppose, moreover, that body 2 turns *clockwise* relative to body 1 about axis 12. Then the vector 23-b, drawn through mutual center 23 perpendicular to 12 - 23 represents the linear velocity, relative to body 1, of point 23, a point in body 2. But, Theorem VI, Property A, point 23 being the velocity-center of bodies 2 and 3, is a point for the instant not only in body 2 but also in body 3; therefore, vector 23-b represents also the velocity, relative to body 1, of point 23 as a point in body 3. But if vector 23-b represents the velocity of center 23 as a point in body 3, body 3 must at the instant be turning *counterclockwise* relative to body 1 about axis 13; that is, bodies 2 and 3 are turning in opposite senses relative to body 1. Moreover, obviously, as long as mutual center 23 lies *anywhere* between centers 12 and 23 bodies 2 and 3 will turn in opposite senses relative to body 1, the body of reference.

So that then we can state the rule of signs for equations (51), (56), and (61) as follows:

(1) If the mutual center lies *anywhere* on the velocity-center axis except between the other two centers the moving bodies turn in the same sense relative to the body of reference, and the sign of the ratio is positive.

(2) If the mutual center lies *anywhere* on the velocity-center axis between the other two centers the moving bodies turn in opposite senses relative to the body of reference, and the sign of the ratio is negative.

Assuming that the signs of equations (51), (56), and (61) are based on Fig. 25b, let us now rewrite these equations, giving to them signs in accordance with the rule of signs just derived.

The sign of the first must, we observe, be negative because the mutual center lies between the other two centers. The sign of each of the other two must be positive because the mutual center in each of these cases does not lie between the other two centers. Therefore if the velocity-centers of the three rigid bodies are positioned relative to each other as shown in Fig. 25b, equations (51), (56), and (61) become

$$\frac{N_{21}}{N_{31}} = -\frac{23 - 31}{23 - 21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (62)$$

$$\frac{N_{12}}{N_{32}} = \frac{13 - 32}{13 - 12}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (63)$$

$$\frac{N_{13}}{N_{23}} = \frac{12 - 23}{12 - 13}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (64)$$

Comments on Theorem VIII.--1. A careful examination of each of the equations (62), (63), and (64) will show the truth of the following very useful statements:

(a) The first symbol in the numerator of the expression for a cubic angular velocity ratio is the same as the symbol that designates the mutual center, and is thus got by combining the first numeral of each subscript on the left-hand side of the equation.

(b) The second symbol in the numerator of the expression is the same as the symbol that designates the secondary center of the ratio, and is thus the same as the subscript of the denominator of the left-hand side of the equation.

(c) The first symbol in the denominator of the expression is the same as the first symbol in the numerator.

(d) The second symbol in the denominator of the expression is the same as the symbol that designates the primary center of the ratio, and is therefore the same as the subscript of the numerator of the left-hand side of the equation.

The four rules just stated make it quite easy to write down the expression for a cubic angular velocity ratio, and are consequently very useful.

2. That two of the three cubic angular velocity ratios must always be positive and one negative follows at once, of course, since only one of the three centers can lie between the other two.

3. In using a cubic angular velocity ratio in solving a problem the student must be very careful, always, to fix the sign of it, because a failure to do so, generally means a serious error in the result.

4. To consider this important matter of signs a little further, let us rearrange the centers for the three rigid bodies of Fig. 25b and then determine the signs of the three associated cubic angular velocity ratios. Arrange the centers as shown in 25c.

For this relative arrangement of the three centers center 12 lies between the other two; but center 12 is the mutual center for ratio N_{13}/N_{23}; therefore for this arrangement of the centers the sign of the right-hand side of equation (64) is negative, and the signs of the other two ratios are positive.

5. Associated with three rigid bodies undergoing instantaneous complanar relative motion at any instant there are, of course, six senses of rotation: two senses for each pair of bodies. It is convenient to have a method of determining five of these senses of rotation when one of them is given.

Suppose body 2, Fig. 25b, to rotate *clockwise* for the instant about axis 12 relative to body 1, let us determine the sense of rotation of body 3 relative to body 1 and the sense of rotation of body 2 relative to body 3. Then, evidently, knowing these three senses of rotation, we shall know all six senses of rotation, because, as we have indicated, the six senses occur in pairs, and the two senses of each pair are opposite to each other.

Bodies 2 and 3 evidently rotate in opposite senses relative to body 1, because center 23, which bodies 2 and 3 have in common, lies between centers 12 and 13. But, by hypothesis, body 2 turns clockwise relative to body 1; therefore body 3 turns counterclockwise relative to body 1, and thus body 1 clockwise relative to body 3. But if body 1 turns clockwise relative to body 3, so also does body 2 turn clockwise relative to body 3, because center 12, which bodies 1 and 2 have in common, does not lie between centers 13 and 23.

To illustrate this procedure still further, let us assume that, Fig. 25b, body 3 turns counterclockwise relative to body 2 and then let us determine the senses of rotation of 2 and 1 and the senses of rotation of 3 and 1.

Bodies 3 and 1 evidently turn in the same sense relative to body 2, because center 13, which bodies 1 and 3 have in common, does not lie between the other two centers, 12 and 23. But, by hypothesis, body 3 turns counterclockwise relative to body 2; therefore body 1 turns counterclockwise relative to body 2, and thus body 2 clockwise relative to body 1. But if body 2 turns clockwise relative to body 1, body 3 turns counterclockwise relative to body 1, because center 23, which bodies 2 and 3 have in common, lies between centers 12 and 13.

6. It is hardly necessary to remark that equations (62), (63), and (64) are not independent of each other, because, for one thing, by the laws of algebra and the fundamental kinematic axiom we can derive any one of them from the other two; indeed, when we know the numerical value and sign of any one of them we can, as we shall presently see, find the numerical value and sign of each of the other two.

To show the truth of the first of these two statements let us derive equation (63) from equations (62) and (64).

Multiplying the left-hand sides of equations (62) and (64) together and putting this product equal to the product of the right-hand sides, we have

$$\frac{N_{21}}{N_{31}} \times \frac{N_{13}}{N_{23}} = -\frac{23 - 31}{23 - 21} \times \frac{12 - 23}{12 - 13}. \quad \ldots\ldots\ldots\ldots \quad (65)$$

But

$$N_{13} = -N_{31}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (66)$$

and

$$23 - 21 = 12 - 23; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (67)$$

therefore

$$-\frac{N_{21}}{N_{31}} \times \frac{N_{31}}{N_{23}} = -\frac{23 - 31}{12 - 13}, \quad \ldots\ldots\ldots\ldots\ldots \quad (68)$$

that is,

$$\frac{N_{21}}{N_{23}} = \frac{23 - 31}{12 - 13}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (69)$$

or since

$$N_{21} = - N_{12}, \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (70)$$

$$N_{23} = - N_{32}, \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (71)$$

$$23 - 31 = 13 - 32, \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (72)$$

and

$$12 - 13 = 13 - 12, \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (73)$$

it follows that

$$\frac{N_{12}}{N_{32}} = \frac{13 - 32}{13 - 12}, \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (74)$$

which is equation (63).

Corollary I to Theorem VIII.--When the numerical value and sign of one of the three cubic angular velocity ratios associated with three rigid bodies undergoing instantaneous complanar relative motions are known, the numerical values and signs of the other two ratios can be calculated.

Proof.--1. Let us suppose in the first place that for three rigid bodies undergoing instantaneous complanar relative motion

$$\frac{N_{21}}{N_{31}} = - m$$

where m is any real negative number whatever. Then the three velocity-centers of the figure must be so arranged that mutual center 23 lies between the other two, because by hypothesis the given ratio is negative. There are two ways in which the centers can be arranged with center 23 between the other two. Figs. 26a and 26b show the two possible arrangements. For the first of these, Theorem VIII gives us

$$\frac{N_{21}}{N_{31}} = - m = - \frac{23 - 31}{23 - 21}, \quad \ldots \ldots \ldots \ldots \quad (75)$$

$$\frac{N_{12}}{N_{32}} = \frac{13 - 32}{13 - 12}, \quad \ldots \ldots \ldots \ldots \quad (76)$$

and

$$\frac{N_{13}}{N_{23}} = \frac{12 - 23}{12 - 13}. \quad \ldots \ldots \ldots \ldots \quad (77)$$

Thus from equation (75) we have

$$\frac{m}{1} = \frac{23 - 31}{23 - 21}, \quad \ldots \ldots \ldots \ldots \quad (78)$$

which shows us that segment 23 - 31, Fig. 26a, is proportional to the number m and segment 23 - 21 to 1. Dimension the figure accordingly and observe that segment 12 - 13 must be proportional to $m + 1$. Therefore from the figure

$$\frac{N_{12}}{N_{32}} = \frac{13 - 32}{13 - 12} = \frac{m}{m + 1}, \quad \ldots \ldots \ldots \ldots \quad (79)$$

and

$$\frac{N_{13}}{N_{23}} = \frac{12 - 23}{12 - 13} = \frac{1}{m + 1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (80)$$

Thus for the first of the two arrangements of centers we get the numerical values $m/(m + 1)$ and $1/(m + 1)$, respectively, for the angular velocity ratios N_{12}/N_{32} and N_{21}/N_{23} when $N_{21}/N_{31} = -m$.

For the second arrangement of centers, Fig. 26b, for which as in the first case $N_{21}/N_{31} = -m$, Theorem VIII gives us again, of course, the same three equations (75), (76), and (77). Thus, just as before

$$\frac{m}{1} = \frac{23 - 31}{23 - 21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (81)$$

Dimensioning Fig. 26b in accordance with this proportion, and reasoning exactly as for the preceding case, we get

$$\frac{N_{12}}{N_{32}} = \frac{13 - 32}{13 - 12} = \frac{m}{m + 1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (82)$$

and

$$\frac{N_{13}}{N_{23}} = \frac{12 - 23}{12 - 13} = \frac{1}{m + 1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (83)$$

the same numerical values for these two ratios as in the preceding case.

2. Let us next suppose that for the same three rigid bodies at the same instant

$$\frac{N_{21}}{N_{31}} = +m,$$

where m is any real positive number whatever.

The three velocity-centers 12, 13, 23 must in this case be so arranged in the same straight line that mutual center 23 does not lie between the other two centers, because, by hypothesis, the given ratio is positive. There are four ways in which the three centers may be arranged with mutual center 23 not between the other two. These four arrangements are as shown in Figs. 26c, 26d, 26e, 26f. And for each one of them Theorem VIII gives us

$$\frac{N_{21}}{N_{31}} = \frac{23 - 31}{23 - 21} = m, \quad \ldots\ldots\ldots\ldots\ldots\ldots (84)$$

that is

$$\frac{23 - 31}{23 - 21} = \frac{m}{1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (85)$$

Dimensioning Figs. 26c-26f in accordance with equation (85), we have

For Fig. 26c

$$\frac{N_{12}}{N_{32}} = \frac{13 - 32}{13 - 12} = \frac{m}{m - 1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (86)$$

$$\frac{N_{13}}{N_{23}} = -\frac{12 - 23}{12 - 13} = -\frac{1}{m - 1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (87)$$

For Fig. 26d

$$\frac{N_{12}}{N_{32}} = -\frac{13-32}{13-12} = -\frac{m}{1-m} \quad \ldots\ldots\ldots\ldots \quad (88)$$

$$\frac{N_{13}}{N_{23}} = \frac{12-23}{12-13} = \frac{1}{1-m} \quad \ldots\ldots\ldots\ldots \quad (89)$$

For Fig. 26e

$$\frac{N_{12}}{N_{32}} = \frac{13-32}{13-12} = \frac{m}{m-1} \quad \ldots\ldots\ldots\ldots \quad (90)$$

$$\frac{N_{13}}{N_{23}} = -\frac{12-23}{12-13} = -\frac{1}{m-1} \quad \ldots\ldots\ldots\ldots \quad (91)$$

For Fig. 26f

$$\frac{N_{12}}{N_{32}} = -\frac{13-32}{13-12} = -\frac{m}{1-m} \quad \ldots\ldots\ldots\ldots \quad (92)$$

$$\frac{N_{13}}{N_{23}} = \frac{12-23}{12-13} = \frac{1}{1-m} \quad \ldots\ldots\ldots\ldots \quad (93)$$

A comparison of equations (86), (88), (90), and (92) shows that they all give exactly the same numerical value and sign for the ratio N_{12}/N_{32}; and, similarly, a comparison of equations (87), (89), (91), and (93) shows that they all give the same numerical value and sign for the ratio N_{13}/N_{23}. Thus we see that the numerical values and signs of the ratios N_{12}/N_{32} and N_{13}/N_{23} are the same however the three velocity-centers may be arranged, provided $N_{21}/N_{31} = m$; provided, that is, mutual center 23 is not between the other two centers and the numerical value m is the same for all four arrangements.

We have now shown that when the sign and numerical value of the ratio N_{21}/N_{31}, one of the three cubic angular velocity ratios associated at any instant with three rigid bodies, 1, 2, 3, undergoing complanar relative motion, are given we can find the sign and numerical value of each of the other two ratios of the set. But, obviously, exactly the same argument that has been used in the case of N_{21}/N_{31} is applicable to each of the other two ratios; thus we conclude that Corollary I to Theorem VIII is true.

Theorem IX.--If three rigid bodies, 1, 2, 3, are undergoing at a given instant complanar motion relative to each other their three angular velocities are connected by the scalar equation

$$N_{21} = N_{31} + N_{23}. \quad \ldots\ldots\ldots\ldots \quad (94)$$

Proof.--Consider any one of the Figs. 26a-26f, say 26d. For this figure Theorem VIII gives the equations

$$\frac{N_{21}}{N_{31}} = \frac{23-31}{23-21}, \quad \ldots\ldots\ldots\ldots \quad (95)$$

$$\frac{N_{12}}{N_{32}} = -\frac{13-32}{13-12}, \quad \ldots\ldots\ldots\ldots \quad (96)$$

$$\frac{N_{13}}{N_{23}} = \frac{12-23}{12-13}; \quad \ldots\ldots\ldots\ldots \quad (97)$$

the sign of the right-hand side of the second of these being negative because the figure shows that mutual center 13 lies between the other two centers and that therefore bodies 1 and 3 turn at the instant in opposite senses relative to the body of reference, body 2 for this ratio N_{12}/N_{32}.

Now invert equations (95) and (96) and set the sum of their left-hand sides thus inverted equal to the sum of their right-hand sides, inverted, and we have

$$\frac{N_{31}}{N_{21}} + \frac{N_{32}}{N_{12}} = \frac{23 - 21}{23 - 31} - \frac{13 - 12}{13 - 32}. \quad \ldots\ldots\ldots\ldots \quad (98)$$

But

$$\frac{N_{32}}{N_{12}} = \frac{-N_{23}}{-N_{21}} = \frac{N_{23}}{N_{21}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (99)$$

because, axiomatically,

$$N_{32} = -N_{23}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (100)$$

and

$$N_{12} = -N_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (101)$$

Also, evidently, from the figure

$$(23 - 31) = (13 - 32). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (102)$$

Therefore we may write equation (98) in the form

$$\frac{N_{31} + N_{23}}{N_{21}} = \frac{(23 - 21) - (13 - 12)}{23 - 31}. \quad \ldots\ldots\ldots\ldots \quad (103)$$

But the figure shows that

$$(23 - 21) - 13 - 12) = (23 - 31); \quad \ldots\ldots\ldots\ldots \quad (104)$$

therefore

$$\frac{(23 - 21) - (13 - 12)}{23 - 31} = 1, \quad \ldots\ldots\ldots\ldots\ldots \quad (105)$$

and consequently also

$$\frac{N_{31} + N_{23}}{N_{21}} = 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (106)$$

that is,

$$N_{21} = N_{31} + N_{23}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (107)$$

which is what was to be proved.

Comments on Theorem IX.--1. (a) This theorem may be symbolically expressed in six different ways with two terms on the right-hand side of the equation, one term on the left-hand side, and all terms positive, as follows:

$$N_{21} = N_{31} + N_{23}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (108$$

$$N_{12} = N_{32} + N_{13}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (109)$$

$$N_{31} = N_{21} + N_{32}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (110)$$

$$N_{13} = N_{23} + N_{12}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (111)$$

$$N_{23} = N_{13} + N_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (112)$$

$$N_{32} = N_{12} + N_{31}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (113)$$

Moreover any one of the five equations (109)-(113) can be proved in precisely the same way that equation (108) was proved. To show this, let us write out the proof of equation (112), for example.

Add, without inverting, the left-hand sides of equations (97) and (96), and set this sum of the right-hand sides of these equations. Thus

$$\frac{N_{13}}{N_{23}} + \frac{N_{12}}{N_{32}} = \frac{12 - 23}{12 - 13} - \frac{13 - 32}{13 - 12}. \quad \ldots\ldots\ldots\ldots \quad (114)$$

Since, however,

$$N_{12} = -N_{21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (115)$$

$$N_{32} = -N_{23}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (116)$$

and

$$12 - 13 = 13 - 12, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (117)$$

we have

$$\frac{N_{13} + N_{21}}{N_{23}} = \frac{(12 - 23) - (13 - 32)}{12 - 13}. \quad \ldots\ldots\ldots\ldots \quad (118)$$

But, from the figure, Fig. 26d,

$$(12 - 23) - (13 - 32) = 12 - 13; \quad \ldots\ldots\ldots\ldots \quad (119)$$

therefore

$$\frac{(12 - 23) - (13 - 32)}{12 - 13} = 1. \quad \ldots\ldots\ldots\ldots\ldots \quad (120)$$

and thus also

$$N_{23} = N_{13} + N_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (121)$$

(b) We can, of course, transform any one of the five equations (109)-(113(into equation (108)--and thus any one of them into any other--by using the algebraic law of transposition and the kinematic axiom to which reference has been made several times.

To show the truth of the statement just made let us convert equation (112) into equation (108), which has been proved.

Transposing N_{13} to the other side of equation (112), we have

$$N_{21} = N_{23} - N_{13}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (122)$$

which, because

$$-N_{13} = N_{31}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (123)$$

is the same as

$$N_{21} = N_{31} + N_{23}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (124)$$

(c) Hence it is, or ought to be, evident that equations (108)-(113) are not six independent equations but that the six of them give us no more information about the relative motions of the bodies at the instant than any one of them does. In solving practical problems, however, one of the six is apt to be much more convenient algebraically than any of the others. That, mainly, is our justification for considering all six forms.

(d) There is another useful form in which Theorem IX may be symbolically expressed, the form, namely, in which all three terms of the equation are positive and on the same side of the equality sign and zero on the other.

Take equation (108), for example, and transpose N_{21} to the other side of it. Then

$$-N_{21} + N_{31} + N_{23} = 0, \quad \ldots\ldots\ldots\ldots \quad (125)$$

that is, since

$$-N_{21} = N_{12},$$

$$N_{12} + N_{31} + N_{23} = 0, \quad \ldots\ldots\ldots\ldots \quad (126)$$

or, changing the order of the last two terms,

$$N_{12} + N_{23} + N_{31} = 0. \quad \ldots\ldots\ldots\ldots \quad (127)$$

2. *The Law of Subscripts for the Equations of Theorem IX.*--In using the equations of Theorem IX in solving practical problems the student is just wasting his time if he does not thoroughly understand how to arrange the subscripts of the terms of these equations, because a mistake in the order of the numerals in a subscript has exactly the same disastrous consequences on the final result as a mistake in sign. Let us, therefore, consider this matter of the order of subscripts very fully and carefully.

(a) If the student will examine the subscripts of equations (108)-(113) he will see that the order of their arrangement obeys the following rules:

i. The first numeral in the subscript of the first term on the right-hand side of the equation is that one of the three numerals that doesn't appear in the subscript of the term on the left-hand side of the equation.

ii. The second numeral in the subscript of the first term on the right-hand side of the equation is the same as the second numeral of the subscript of the term on the left-hand side of the equation.

iii. The first numeral in the subscript of the second term on the right-hand side of the equation is the same as the first numeral of the subscript of the term on the left-hand side of the equation.

iv. The second numeral in the subscript of the second term on the right-hand side of the equation is the same as the first numeral of the subscript of the first term on the right-hand side of the equation.

(b) Consider next the law of subscripts for Theorem IX when this theorem is expressed in the form of equation (127).

Draw, freehand, a small circle, called the cyclic circle, Fig. 27, and symbolize it as shown, with the numerals 1, 2, 3, if the rigid bodies under consideration are so numbered. These numerals, as the student will presently see, may be placed in either a clockwise or a counterclockwise order around the circle. Now start at any one of the numbers and go around the circle in either sense. Let us start, for example, at 3 and go counterclockwise until we are back at the number 3 again. Thus, proceeding from 3 to 2, from 2 to 1, from 1 to 3, whence we started, we get

$$N_{32} + N_{21} + N_{13} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (128)$$

or changing as, of course, we may, the signs throughout,

$$-N_{32} - N_{21} - N_{13} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (129)$$

that is,--since axiomatically

$$-N_{32} = N_{23},$$

$$-N_{21} = N_{12},$$

$$-N_{13} = N_{31}. --$$

$$N_{23} + N_{12} + N_{31} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (130)$$

which, except for the immaterial order of the terms, is the same as equation (126),

Secondly, start first at 2, and then at 1 and go counterclockwise in both cases until the cycle is complete, and we get in the first case

$$N_{21} + N_{13} + N_{32} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (131)$$

and in the second

$$N_{13} + N_{32} + N_{21} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (132)$$

Now comparing equations (128), (131), and (132), we see that they are exactly alike except for the order in which the three terms are added, and since the order in which the terms are added is of no consequence, we conclude that in using the cyclic rule to fix the order of the subscripts for equation (126) we may start with whichever one of the numerals we please, provided we always go in a counterclockwise sense around the circle.

To show that the sense in which we go around the circle doesn't affect the result either, let us start again at 3, Fig. 27, and go *clockwise* around the circle. Then we get

$$N_{31} + N_{12} + N_{23} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (133)$$

or, changing the signs throughout,

$$-N_{31} - N_{12} - N_{23} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (134)$$

But

$$-N_{31} = N_{13},$$

$$-N_{12} = N_{21},$$

$$-N_{23} = N_{32};$$

hence

$$N_{13} + N_{21} + N_{32} = 0;$$

thus equation (133) is transformed into equation (132), which we know to be true. Now, finally, let us see that the order in which the numerals are placed on the circle doesn't matter.

Place the numerals in counterclockwise order, Fig. 28, and start at any one of them, say at 1, and go around the circle in either sense, say counterclockwise. Then we have

$$N_{12} + N_{23} + N_{31} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (135)$$

which is the same as equation (127).

Thus we conclude that in using the cyclic rule to arrange the subscripts in such equations as (128), (130), and (135) in the proper order we may (1) put the numerals on the circle in either order, (2) start at any numeral, and (3) go in either sense around the circle.

The Complanar Relative Motions of Four Rigid Bodies.--Theorems VIII and IX deal, as the student has seen, with the complanar relative motions of *three* rigid bodies; the theorem whose consideration we are now beginning is concerned with the complanar relative motions of *four* rigid bodies.

When four rigid bodies undergo instantaneous complanar relative motion, we have six cases of simultaneous rotation, because according to Theorem VI we have a case of rotation for each pair of bodies and the number of pairs of bodies is six. To see the truth of this last statement the student must remember that the theory of *Permutations and Combinations* shows us that the number of combinations of n things taken two at a time is $n(n - 1)/2$, so that if, as is the case here, n is 4, $n(n - 1)/2$ is 6.

Thus, moreover, we have at each instant of the motion of the four bodies six velocity-axes or centers, which, if the bodies are numbered 1, 2, 3, 4, are as follows, when arranged in tabular form:

12
13 23
14 24 34

For a quartic angular velocity ratio--for the quartic ratio N_{21}/N_{34}, for example--the six corresponding velocity-centers are called a *center-sextet* and are designated as follows:

(1) The subscript of the numerator of the ratio, 21 or 12, is known as the *First* center.

(2) The subscript of the denominator, 34 or 43, as the *Second* center.

(3) The symbol, 23 or 32, derived by combining the first numeral of the upper subscript with the first numeral of the lower subscript is called the *Third* center.

(4) The symbol, 14 or 41, derived by combining the second numerals of the two subscripts, is the *Fourth* center.

(5) The symbol, 24 or 42, derived by combining the first numeral of the upper subscript with the second numeral of the lower subscript is called the *Fifth* center.

(6) The symbol, 13 or 31, derived by combining the second numeral of the upper subscript with the first numeral of the lower subscript designates the *Sixth* center.

In illustration of these definitions let us apply them to the other two quartic angular velocity ratios of the set

$$\frac{N_{21}}{N_{34}}, \quad \frac{N_{14}}{N_{23}}, \quad \frac{N_{31}}{N_{24}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (C)$$

For the second quartic angular velocity ratio, N_{14}/N_{23}, of this set, the same six centers at the same instant are designated, according to the preceding definitions, as follows:

14 or 41 First Center
23 or 32 Second "
12 or 21 Third "
34 or 43 Fourth "
13 or 31 Fifth "
24 or 42 Sixth "

And for the third quartic ratio, N_{31}/N_{24}, of Set (C) the centers are:

31 or 13 First Center
24 or 41 Second "
23 or 32 Third "
14 or 41 Fourth "
34 or 43 Fifth "
12 or 21 Sixth "

Since, in the next place, there are four bodies in the system there are four groups of bodies with three bodies in each group. This we know because--if a reason is necessary --from the theory of *Permutations and Combinations*, the number of combinations of n things taken three at a time is

$$\frac{n(n-1)(n-2)}{1 \times 2 \times 3},$$

which when n is 4 takes the value 4.

The four groups of the four bodies with three bodies in each group are, if the bodies are numbered 1, 2, 3, 4,

123, 124, 134, 234.

Moreover, to each of these groups of three bodies belong three centers, which according to Theorem VII, the *Kennedy-Aronhold Theorem*, must, at each and every instant of the motion, lie on the same straight line. It follows, therefore, that the six centers that belong to four rigid bodies must lie in sets of three on four straight lines.

To illustrate these statements consider Fig. 29a, which represents a set of six velocity-centers, that is, a *centersextet*, for four rigid bodies undergoing instantaneous complanar motion, arranged in accordance with the principles just explained.

Let the student carefully note that the six centers of the sextet are arranged in sets of three on four straight lines, L_1, L_2, L_3, and L_4, respectively. These lines, the student will remember, are called velocity-center axes. Let him note also: (1) that each such set of three centers involves three and only three of the four bodies in the system, (2) that each center of each set of three centers belongs to one other set of three centers. Take, for example, the set of three centers, 24, 12, 14. These three centers involve only the bodies 1, 2, and 4, and each center belongs to one other set of three centers: center 24 to set 24, 23, 34, center 12 to set 12, 23, 13, and center 14 to set 14, 34, 13.

Besides the lines L_1, L_2, L_3, L_4, on which lie the four sets of three centers, there are three other lines in the figure D_1, D_2, and D_3. These three lines are called diagonals of the complete quadrilateral whose sides are lines L_1, L_2, L_3, L_4.

Diagonal D_1, called the *first diagonal*, joins the *First* to the *Second* center of the sextet; D_2, called the second diagonal, joins the *Third* center to the *Fourth;* and D_3, called the third diagonal, joins the *Fifth* and *Sixth* centers. Thus, Fig. 29a, for the quartic angular velocity ratio N_{21}/N_{34}, D_1 is 21-34, D_2 is 23-14, and D_3 is 24-13.

The points of intersection of diagonals D_1 and D_2, of diagonals D_1 and D_3, and of diagonals D_2 and D_3 are designated with the symbols O_{12} or O_{21}, O_{13} or O_{31}, O_{23} or O_{32}, respectively, as shown in Fig. 29a. Moreover, these three points, respectively, are called the First, Second, and Third diagonal points of the centersextet.

Diagonals and sides of the quadrilateral are distinguished from each other by the fact that each of the diagonals, D_1, D_2, and D_3, joins only *two* of the six centers and and these two centers involve all four of the bodies, whereas each of the lines L_1, L_2, L_3, and L_4, contains *three* centers that involve only three of the four bodies.

We are now prepared to state and prove Theorem X.

Theorem X.--(a) The $\left\{\begin{matrix}\text{second}\\ \text{third}\end{matrix}\right\}$ diagonal of a quartic angular velocity ratio divides the segment between the first and second centers of the ratio into two segments whose inverse ratio is equal to the quartic ratio in question.

(b) Moreover, the sign of the ratio is *positive* if the Second Diagonal divides the segment between the First and Second centers *externally* and the Third diagonal divides this segment *internally;* the sign of the ratio is *negative* if the Second diagonal divides the segment between the First and Second centers *internally* and the *Third* diagonal divides it *externally.*

Let the four rigid bodies of the ratio be numbered 1, 2, 3, and 4, and let the centersextet associated with the four bodies at a given instant be arranged as shown in Fig. 29a. Then we are to prove that

$$\frac{N_{21}}{N_{34}} = \pm\frac{O_{12}-34}{O_{12}-21} = \pm\frac{O_{13}-34}{O_{13}-21}. \quad \ldots\ldots\ldots\ldots \quad (136)$$

Proof.--A. The quartic ratio, N_{21}/N_{34}, can be expressed as the product of two cubic ratios in four ways, as follows:

$$\frac{N_{21}}{N_{34}} = \frac{N_{21}}{N_{23}}\times\frac{N_{23}}{N_{34}} = -\frac{N_{12}}{N_{32}}\times\frac{N_{23}}{N_{43}} = -\frac{13-32}{13-12}\times\frac{24-43}{24-23}, \quad \ldots\ldots \quad (137)$$

$$\frac{N_{21}}{N_{34}} = \frac{N_{21}}{N_{31}}\times\frac{N_{31}}{N_{34}} = \frac{N_{21}}{N_{31}}\times\frac{N_{13}}{N_{43}} = -\frac{23-31}{23-21}\times\frac{14-43}{14-13}, \quad \ldots\ldots\ldots \quad (138)$$

$$\frac{N_{21}}{N_{34}} = \frac{N_{21}}{N_{24}}\times\frac{N_{24}}{N_{34}} = \frac{N_{12}}{N_{42}}\times\frac{N_{24}}{N_{34}} = -\frac{14-42}{14-12}\times\frac{23-34}{23-24}, \quad \ldots\ldots\ldots \quad (139)$$

$$\frac{N_{21}}{N_{34}} = \frac{N_{21}}{N_{41}}\times\frac{N_{41}}{N_{34}} = -\frac{N_{21}}{N_{41}}\times\frac{N_{14}}{N_{34}} = -\frac{24-41}{24-21}\times\frac{13-34}{13-14}, \quad \ldots\ldots\ldots \quad (140)$$

These four equations hold for the quartic angular velocity ratio N_{21}/N_{34} however the centersextet associated with the ratio may be arranged, but of course the final signs of their right-hand sides will not be the same for all arrangements of the centers. For some arrangements, in other words, these signs will be positive; for other arrangements they will be negative. For the arrangement given by Fig. 29a, for example,

(a) The sign of the right-hand side of equation (137) is *negative* because, Theorem VIII, Convention of Signs, the signs of the two cubic ratios N_{12}/N_{32} and N_{23}/N_{43} are both positive.

(b) The sign of the right-hand side of equation (138) is *negative* because the sign of the cubic ratio N_{21}/N_{31} is *negative* and the sign of the cubic ratio N_{13}/N_{43} is *positive*,

(c) The sign of the right-hand side of equation (139) is *negative* because the sign of the cubic ratio N_{12}/N_{34} is *positive* while the sign of the cubic ratio N_{24}/N_{34} is *negative*.

(d) The sign of the right-hand side of equation (140) is *negative* because the signs of the two cubic ratios N_{21}/N_{41} and N_{14}/N_{34} are both positive.

Hence all four equations when applied to Fig. 29a give the same sign, the negative sign, for the quartic ratio N_{21}/N_{34}.

These equations, their signs fixed by the conditions of Fig. 29a, in which diagonals D_2 and D_3 divide the segment 12-34 *internally*, and *externally*, respectively, are therefore as follows:

$$\frac{N_{21}}{N_{34}} = -\frac{13-32}{13-12}\times\frac{24-43}{24-23}, \quad \ldots\ldots\ldots\ldots \quad (137')$$

$$\frac{N_{21}}{N_{34}} = -\frac{23-31}{23-21}\times\frac{14-43}{14-13}, \quad \ldots\ldots\ldots\ldots \quad (138')$$

$$\frac{N_{21}}{N_{34}} = -\frac{14-42}{14-12}\times\frac{23-34}{23-24}, \quad \ldots\ldots\ldots\ldots \quad (139')$$

$$\frac{N_{21}}{N_{34}} = -\frac{24-41}{24-21}\times\frac{13-34}{13-14}, \quad \ldots\ldots\ldots\ldots \quad (140')$$

1. Now by *Ceva's geometrical Theorem applied to triangle 12, 23, 34, using as pole 14 we have

$$\frac{23-13}{13-12}\times\frac{12-O_{12}}{O_{12}-34}\times\frac{34-24}{24-23} = 1, \quad \ldots\ldots\ldots\ldots \quad (141)$$

that is,

$$\frac{O_{12}-34}{12-O_{12}} = \frac{23-13}{13-12}\times\frac{34-24}{24-23}, \quad \ldots\ldots\ldots\ldots \quad (142)$$

$$= \frac{13-32}{13-12}\times\frac{24-43}{24-23}; \quad \ldots\ldots\ldots\ldots \quad (143)$$

thus, equation (137')

$$\frac{N_{21}}{N_{34}} = -\frac{O_{12}-34}{12-O_{12}}. \quad \ldots\ldots\ldots\ldots \quad (144)$$

2. Applying Ceva's Theorem to triangle 12, 13, 34, using pole 24 we have

$$\frac{13-23}{23-12}\times\frac{12-O_{13}}{O_{13}-34}\times\frac{34-14}{14-13} = 1, \quad \ldots\ldots\ldots\ldots \quad (145)$$

that is,

$$\frac{O_{13}-34}{12-O_{13}} = \frac{13-23}{23-12}\times\frac{34-14}{14-13}, \quad \ldots\ldots\ldots\ldots \quad (146)$$

$$= \frac{23-31}{23-21}\times\frac{14-43}{14-13}; \quad \ldots\ldots\ldots\ldots \quad (147)$$

*See College Geometry by Altshiller-Court, p. 236, or Projective Geometry by Cremona, p. 111.

hence, equation (138')

$$\frac{N_{21}}{N_{34}} = -\frac{O_{13} - 34}{12 - O_{13}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (148)$$

3. Applying Ceva's Theorem to triangle 12, 24, 34 and pole 13, we have

$$\frac{24 - 14}{14 - 12} \times \frac{12 - O_{13}}{O_{13} - 34} \times \frac{34 - 23}{23 - 24} = 1. \quad \ldots\ldots\ldots\ldots (149)$$

that is,

$$\frac{O_{13} - 34}{12 - O_{13}} = \frac{24 - 14}{14 - 12} \times \frac{34 - 23}{23 - 24}, \quad \ldots\ldots\ldots\ldots (150)$$

$$= \frac{14 - 42}{14 - 12} \times \frac{23 - 34}{23 - 24}; \quad \ldots\ldots\ldots\ldots (151)$$

hence, equation (139')

$$\frac{N_{21}}{N_{34}} = -\frac{O_{13} - 34}{12 - O_{13}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (152)$$

4. Ceva's Theorem applied to triangle 12, 14, 34, and pole 23 gives

$$\frac{14 - 24}{24 - 12} \times \frac{12 - O_{12}}{O_{12} - 34} \times \frac{34 - 13}{13 - 14} = 1, \quad \ldots\ldots\ldots\ldots (153)$$

that is,

$$\frac{O_{12} - 34}{12 - O_{12}} = \frac{14 - 24}{24 - 12} \times \frac{34 - 13}{13 - 14}. \quad \ldots\ldots\ldots\ldots (154)$$

$$= \frac{24 - 41}{24 - 21} \times \frac{13 - 34}{13 - 14}; \quad \ldots\ldots\ldots\ldots (155)$$

hence, equation (140')

$$\frac{N_{21}}{N_{34}} = -\frac{O_{12} - 34}{12 - O_{12}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (156)$$

5. Fixing the signs of the right-hand sides of equations (137), (138), (139), and (140)--which are general and hold for any centersextet--in accordance with the centersextet of Fig. 29b, in which diagonals D_2 and D_3 divide segment 12- 34 *externally* and *internally*, we get

$$\frac{N_{21}}{N_{34}} = \frac{13 - 32}{13 - 12} \times \frac{24 - 43}{24 - 23}, \quad \ldots\ldots\ldots\ldots (137'')$$

$$\frac{N_{21}}{N_{34}} = \frac{23 - 31}{23 - 21} \times \frac{14 - 43}{14 - 13}, \quad \ldots\ldots\ldots\ldots (138'')$$

$$\frac{N_{21}}{N_{34}} = \frac{14 - 42}{14 - 12} \times \frac{23 - 34}{23 - 24}, \quad \ldots\ldots\ldots\ldots (139'')$$

$$\frac{N_{21}}{N_{34}} = \frac{24 - 41}{24 - 21} \times \frac{13 - 34}{13 - 14}. \quad \ldots\ldots\ldots\ldots (140'')$$

Applying Ceva's Theorem four times in succession to the following triangles and corresponding poles of Fig. 29b: (1) to triangle 12, 23, 34 and pole 14, (2) to triangle 12, 13, 34 and pole 24, (3) to triangle 12, 24, 34 and pole 13, (4) to triangle 12, 14, 34 and pole 23, and then combining the four equations thus derived with equations (137"), (138"), (139"), (140"), respectively, we obtain

$$\frac{N_{21}}{N_{34}} = \frac{O_{12} - 34}{12 - O_{12}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (157)$$

$$\frac{N_{21}}{N_{34}} = \frac{O_{13} - 34}{12 - O_{13}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (158)$$

$$\frac{N_{21}}{N_{34}} = \frac{O_{13} - 34}{12 - O_{13}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (159)$$

$$\frac{N_{21}}{N_{34}} = \frac{O_{12} - 34}{12 - O_{12}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (160)$$

which except for a difference in sign, are exactly the same as (144), (148), (152), and (156) for the centersextet of Fig. 29a.

6. Of these eight equations, 144, 148, 152, 156, 157, 158, 159, 160, equations (144), (156), (157), and (160) show that the *second diagonal* D_2 of the two centersextets divides the segment between the first and second centers 21 and 34 in accordance with the first part of Theorem X. And equations (148), (152), (158), and (159) show that the *third diagonal* D_3 divides this segment in accordance with the first part of Theorem X. Thus the first part of the theorem is true.

In the centersextet of Fig. 29b the second diagonal D_2 divides the segment between the first and second centers *externally* and the third diagonal divides this segment *internally;* thus to agree with the second part of Theorem X the sign of quartic ratio N_{21}/N_{34} should be *positive;* equations (157)-(160) show that this is the case. In the centersextet of Fig. 29a the second diagonal D_2 divides the segment between the first and second centers *internally* and the third diagonal D_3 divides this segment *externally;* thus to agree with the second part of Theorem X the sign of quartic ratio N_{21}/N_{34} should be *negative;* equations (144), (148), (152), and (156) prove this to be the case. Thus the second part of the theorem is true.

Thus the theorem is true.

Let us now apply this Theorem X to the solution of a practical problem.

Problem 10.--For the lever-crank linkage of Fig. 29c given: (1) the lengths of the links; (2) the configuration of the linkage ($\theta = 150^{\circ}$); (3) the angular speed of crank 2 ($N_{21} = 600$ *rpm.*); (4) the radius of the bearing between lever 4 and connecting-rod 3 ($R_4 = 1.5"/12$); required the rubbing-speed in fpm. at this bearing for the given configuration.

Solution.--To solve the problem we must evidently use the formula

$$P_3V_{34} = 2\pi N_{34}R_4. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (c)$$

in which P_3V_{34} represents the rubbing-speed in question, N_{34} the angular velocity in rpm. of the connecting-rod relative to the lever, and R_4 in feet the given radius of the bearing at joint 34.

Having determined centers 12, 23, 14, 13, and 24 of the centersextet associated with the linkage and having drawn diagonals D_1 and D_2, to find N_{34} we may, according to Theorem X, use any one of the four formulas 137', 138', 139', and 140', p. 39, each of which

expresses N_{34} in terms of four known linear segments and the given angular velocity N_{21}. Or we may use either of the formulas (144) and (152), pp. 39 and 40, both of which express N_{34} in terms of two known linear segments and N_{21}.

(a) Thus from formula (137'), measuring linear segments 13 - 32, 13 - 12, 24 - 43, and 24 - 23 in hundredths of an inch, we have

$$\frac{N_{21}}{N_{34}} = -\frac{13 - 32}{13 - 12} \times \frac{24 - 43}{24 - 23} = -\frac{291}{391} \times \frac{452}{52}$$

$$= -6.47 \text{ approx.},$$

from which, since $N_{21} = 600$ rpm.,

$$N_{34} = 92.25, \text{ approx.};$$

and hence from formula (c)

$$P_3V_{34} = 2\pi N_{34}R_4 = 6.28 \times 92.25 \times \frac{1.5}{12}$$

$$= 72.41 \text{ fpm.}$$

(b) Or using formula (144), and again measuring segments in hundredths of an inch, we have

$$\frac{N_{21}}{N_{34}} = -\frac{O_{12} - 34}{12 - O_{12}} = -\frac{4130}{635}$$

$$= -6.5 \text{ approx.},$$

from which

$$N_{34} = 92.3 \text{ approx.},$$

a value for N_{34} approximately equal to that obtained by using formula (137'). And this value of N_{34} substituted in formula (c) gives about the same rubbing-speed as that found by the other method.

II. Graphic Determination of Linear Velocities In Mechanisms

Any discussion of this subject logically begins with the statement and proof of the three theorems on which linear velocity determinations depend, viz: (1) the *Theorem of the Triangle of Velocities*, (2) *Rankine's Theorem of Longitudinal Components*, (3) *Burmester's Theorem of Velocity Images*. To these important theorems, which we shall call Theorems XI, XII, and XIV, we now give our attention.

Postulate I.--A linear velocity is perfectly represented by a straight-line segment, if: (1) the *number* of linear units of length in the segment is equal to the number of units of velocity in the velocity in question; (2) the *direction* of the segment is the same as the *direction* of the velocity; (3) the *sense* of the segment, designated usually by an arrowhead, is the same as the sense of the velocity. If these three conditions are met the position in space of the line-representative of the velocity is a matter of indifference; it may be taken in whatever position is most convenient.

Postulate II.--To specify adequately the linear velocity of a point, five data must be given: (1) the rigid body which carries the point; (2) the rigid body of reference in which the point describes its path; (3) the *magnitude* of the linear velocity; (4) the *direction* of the velocity; (5) the *sense* of the velocity.

Postulate III.--To specify adequately the angular velocity of a rigid body, four data must be given: (1) the rigid body of reference in which the points of the moving body describe their paths; (2) the position of the velocity-axis of the moving body and the body of reference in each of the bodies; (3) the *magnitude* of the angular velocity relative to the body of reference; (4) the *sense* of the angular velocity relative to the body of reference.

Axiom I.--If P_2 and P_3 are two instantaneously coincident points in rigid bodies 2 and 3, respectively, then

$$*\,{}_{P_2}V_{23} = -{}_{P_3}V_{32}, \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

where ${}_{P_2}V_{23}$ is the linear velocity of point P_2 in body 2 relative to body 3, and ${}_{P_3}V_{32}$ is the linear velocity of point P_3 in body 3 relative to body 2. Similarly

$${}_{P_2}V_{21} = -{}_{P_1}V_{12}. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

and

$${}_{P_1}V_{13} = -{}_{P_3}V_{31}. \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

Corollary to Theorem VI.--If at and for a given instant three rigid bodies, numbered 1, 2, 3, respectively, are undergoing complanar relative motion, then

$${}_{23}V_{21} = {}_{23}V_{31}, \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

$${}_{12}V_{23} = {}_{12}V_{13}, \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

$${}_{13}V_{12} = {}_{13}V_{32}, \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

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*In this and in all such symbols V is the initial letter of the word velocity; the subscript on the left of V designates the point whose linear velocity is represented by the symbol taken as a whole; the first numeral in the subscript on the right of V designates the body which carries the point whose linear velocity is represented by the symbol taken as a whole; the second numeral in this right-hand subscript designates the rigid body or frame of reference.

where ${}_{23}V_{21}$ is the linear velocity of velocity-center 23 in body 2 relative to body 1, ${}_{23}V_{31}$ the linear velocity of velocity-center 23 in body 3 relative to body 1, and similarly for the other equations.

Lemma I.--Let P_2 and Q_2, Fig. 30, be two points in a rigid body 2 which at and for a given instant is undergoing complanar rotation about velocity axis 12 relative to rigid body 1, and let the linear segment P_2m, laid off along $P_2 - 12$, represent the *magnitude* of the velocity of point P_2. Through point m draw mn parallel to P_2Q_2; then segment Q_2n represents the magnitude of the linear velocity of point Q_2 to the same scale that P_2m represents the magnitude of the linear velocity of point P_2.

Let P_2V_{21} and Q_2V_{21} represent respectively the magnitudes of the linear velocities of points P_2 and Q_2 relative to body 1; then the proposition to be proved, symbolically expressed, is

$$\frac{Q_2V_{21}}{P_2V_{21}} = \frac{Q_2n}{P_2m}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Proof: Since points P_2 and Q_2 are points in the same rigid body, their linear velocities are to each other directly as their respective distances from the center of rotation 12; hence, Theorem VI, Property B

$$\frac{Q_2V_{21}}{P_2V_{21}} = \frac{Q_2 - 12}{P_2 - 12}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

but from the geometry of the figure

$$\frac{Q_2 - 12}{P_2 - 12} = \frac{Q_2n}{P_2m}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

therefore

$$\frac{Q_2V_{21}}{P_2V_{21}} = \frac{Q_2n}{P_2m}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

which obviously asserts that to whatever scale P_2m equals P_2V_{21}, to the same scale Q_2n equals Q_2V_{21}.

Note: This simple lemma plays a very important role in the proofs of the theorem with which we are concerned; it should therefore be carefully noted.

The sole purpose of everything that has been said up to this point is to refresh the student's mind with the necessary fundamental kinematic principles upon which the proofs of our theorems depend.

Now let the three irregular contours of Fig. 31 represent three rigid bodies, 1, 2, 3; let these bodies be undergoing complanar relative motion at and for an instant; let the three *collinear* points 12, 23, and 13 be the three velocity-centers or velocity-axes of these three pairs of bodies for the instant under consideration, Theorem VII; let P_1, P_2, P_3 designate three discrete but instantaneously coincident points in these three bodies, respectively--P_1 in body 1, P_2 in body 2, and P_3 in body 3. Draw the triangle $P_1b'c'$ with its sides respectively perpendicular to lines $P_1 - 12$, $P_2 - 23$, and $P_3 - 13$. Then this theorem may be concisely stated as follows:

Theorem XI.--*The Theorem of the Triangle, or Parallelogram, of Velocities.* If side P_2b' of triangle $P_1b'c'$ represents to scale P_2V_{21}, Fig. 31, that is, represents the linear velocity of point P_2 in body 2 relative to body 1; then sides P_3c' and $c'b'$ to the same scale represent P_3V_{31} and P_2V_{23}, respectively; that is to say, P_3c' represents the linear velocity of point P_3 in body 3 relative to body 1 and side $c'b'$, to the same scale, the linear velocity of point P_2 in body 2 relative to body 3.

Proof of Theorem XI.--About its vertex P_1 rotate triangle $P_1b'c'$ until its side P_1b' coincides with line $P_1 - 12$. This rotation of triangle $P_1b'c'$ obviously brings side P_1c' into coincidence with line $P_3 - 13$ and side $b'c'$ into parallelism with $P_2 - 23$. Let the symbols P_1, b, and c represent the vertices of the triangle in its rotated position. Now produce side bc to cut $12 - 23$ at a; draw cd parallel to $P_1 - 12$, and produce it to cut $23 - 12$ at d and $P_2 - 23$ at e. Then, evidently, triangles P_1bc and P_1ec are congruent.

Now if segment P_2b represents the magnitude of $_{P_2}V_{21}$, then, Lemma I, segment $23 - a$ represents the magnitude of $_{23}V_{21}$; but, Theorem VI, Property A, $_{23}V_{21} = {}_{23}V_{31}$, since point 23 is the velocity-center of bodies 2 and 3; therefore segment $23 - a$ represents also the the magnitude of $_{23}V_{31}$. But if $23 - a$ represents the magnitude of $_{23}V_{31}$, then, Lemma I, segment P_3c represents the magnitude of $_{P_3}V_{31}$; but, Axiom I, $_{P_3}V_{31} = -{}_{P_1}V_{13}$; therefore segment P_3c represents the magnitude of $_{P_1}V_{13}$; but if P_3c represents the magnitude of $_{P_1}V_{13}$, then, Lemma I, segment $12 - d$ represents the magnitude of $_{12}V_{13}$; but, Theorem VI, Property A, $_{12}V_{13} = {}_{12}V_{23}$; therefore segment $12 - d$ represents $_{12}V_{23}$; but if segment $12 - d$ represents the magnitude of $_{12}V_{23}$, then, Lemma I, segment P_2e represents the magnitude of $_{P_2}V_{23}$.

Thus sides P_2b, P_3c, $cb(=P_2e)$ of triangle P_1bc represent, respectively, the magnitudes of linear velocities $_{P_2}V_{21}$, $_{P_3}V_{31}$, and $_{P_2}V_{23}$. But triangles P_1bc and $P_1b'c'$ are congruent; therefore P_2b', P_3c', and $c'b'$ of triangle $P_1b'c'$ represent, respectively, the *magnitudes* of the linear velocities $_{P_2}V_{21}$, $_{P_3}V_{31}$, and $_{P_2}V_{23}$. Since, however, sides P_2b', P_3c', and $c'b'$ of triangle $P_1b'c'$ are at right angles to the lines $P_2 - 12$, $P_3 - 13$, $P_2 - 23$, respectively, these sides are in the right directions to represent correctly the *directions* of the velocities $_{P_2}V_{21}$, $_{P_3}V_{31}$, and $_{P_2}V_{23}$. To conclude the proof, therefore, it must be shown that the arrowheads on the sides of triangle $P_1b'c'$ give the right *senses* of velocities $_{P_2}V_{21}$, $_{P_3}V_{31}$, and $_{P_2}V_{23}$.

To show this, let body 2 rotate clockwise relative to body 1 about velocity-axis axis 12. Then because velocity-center 23 lies between velocity-centers 12 and 13, body 3 rotates counterclockwise about velocity-center 13 relative to body 1. Thus it follows that the arrowheads on sides P_2b' and P_3c' of velocity triangle $P_1b'c'$ give the right senses for $_{P_2}V_{21}$ and $_{P_3}V_{31}$. Again, if body 3 rotates counterclockwise about velocity-axis 13 relative to body 1, body 1 turns clockwise relative to body 3; and if body 1 turns clockwise relative to body 3, body 2 also turns clockwise about axis 23 relative to body 3, because velocity-center 12 does not lie between velocity-centers 23 and 13. But if body 2 turns clockwise relative to body 3, the arrowhead on side $c'b'$ must point, as it does, from c' towards b', to represent correctly the sense of $_{P_2}V_{23}$.

Therefore the truth of Theorem XI becomes fully evident.

Thus we have, moreover, the vector equation

$$_{P_2}V_{21} = {}_{P_3}V_{31} \twoheadrightarrow {}_{P_2}V_{23}, \quad \ldots\ldots\ldots\ldots \quad (11)$$

because this equation asserts merely what we have just proved, viz., that the three linear velocities $_{P_2}V_{21}$, $_{P_3}V_{31}$, and $_{P_2}V_{23}$ are the sides of a triangle, the directions of whose sides are fixed.

Using Axiom I, equation (11) may evidently be written in five other ways, keeping all of the terms positive and one term on one side and two terms on the opposite side of the equation, as follows:

$$_{P_2}V_{23} = {}_{P_1}V_{13} \twoheadrightarrow {}_{P_2}V_{21}, \quad \ldots\ldots\ldots\ldots \quad (12)$$

$$_{P_1}V_{12} = {}_{P_3}V_{32} \twoheadrightarrow {}_{P_1}V_{13}, \quad \ldots\ldots\ldots\ldots \quad (13)$$

$$_{P_1}V_{13} = {}_{P_2}V_{23} \twoheadrightarrow {}_{P_1}V_{12}, \quad \ldots\ldots\ldots\ldots \quad (14)$$

$$_{P_3}V_{31} = {}_{P_2}V_{21} \twoheadrightarrow {}_{P_3}V_{32}, \quad \ldots\ldots\ldots\ldots \quad (15)$$

$$_{P_3}V_{32} = {}_{P_1}V_{12} \twoheadrightarrow {}_{P_3}V_{31}, \quad \ldots\ldots\ldots\ldots \quad (16)$$

If we keep the relative motions of the three rigid bodies of Fig. 31 what they were assumed to be in establishing equation (11), it is easy to determine the senses in which the sides of velocity-triangle $P_1b'c'$ must be read in order that the sides of this triangle may geometrically represent the three terms of any one of the equations (12)-(16). Take equation (14), for example. Since, as we have seen, body 1 turns clockwise relative to body 3, the arrowhead on side P_1c' of velocity-triangle $P_1b'c'$ must point from c' towards P_1 to give the sense of P_1V_{13}. Since, moreover, body 2 turns clockwise relative to body 3 and body 1 turns counterclockwise relative to body 2, the arrowhead on side $c'b'$ must point from c' towards b' to give the sense of P_2V_{23}, and the arrowhead on side $b'P_2$ must point from b' towards P_2 to give the sense of P_1V_{12}.

Rule of Senses.--If the reader will determine the senses of the vectors that represent the two terms on the right-hand side of any one of the equations (11)-(16) in the way just explained he will find in each case that the arrowheads indicating the senses of these two vectors point around the triangle in the opposite sense to the sense in which the arrowhead that indicates the sense of the vector that represents the single term on the left-hand side of the equation. Take, for example, equation (15). The arrowhead on the vector that represents P_3V_{31} points around the triangle *counterclockwise* while the arrowheads on the vectors that represent P_2V_{21} and P_3V_{32} point around the triangle *clockwise.*

It follows therefore that if we know the sense of the vector that represents the term on the left-hand side of any one of equations (11)-(16) we can determine the sense of each of the two vectors that represent the two terms on the right-hand side of the equation.

It should be particularly noted that to each side of velocity-triangle $P_1b'c'$ two rigid bodies correspond, as follows: To side P_1b', bodies 1 and 2; to side P_1c', bodies 1 and 3; to side $c'b'$, bodies 2 and 3.

Corollary I, Theorem XI.--Whenever, therefore, in the complanar kinematics of rigid bodies a velocity-triangle arises, the presence of three rigid bodies is invariably implied.

There is no escape from or exception to this conclusion, and its recognition throws a flood of light on many kinematic obscurities.

Given a Velocity Triangle to Find the Three corresponding Velocity Centers and the Three Corresponding Angular Velocities.--Let us now see how, when we have the velocity-triangle, we can find the three corresponding rigid bodies. Finding the bodies is merely, of course, a matter of finding their velocity-centers and their relative angular velocities. Let $P_1b'c'$, Fig. 31, be the given velocity-triangle, and let the three rigid bodies be numbered 1, 2, 3. Then, through any vertex of triangle $P_1b'c'$, as for example, vertex P_1, draw three lines respectively at right angles to the sides of the triangle. In the figure these lines are P_1b, P_1c, and P_1e. Then the points in which a fourth line, ad, drawn anywhere in the plane of the triangle, except through point P_1 cuts P_1b, P_1c, and P_1e will be the three velocity-centers, 12, 23, and 13, of the three pairs of rigid bodies. In designating the points in which line ad cuts lines P_1b, P_1c, and P_1e, any one of the points of intersection may be designated with any one of the symbols 12, 23, 13; then either of the two remaining points of intersection may be designated with either of the two remaining symbols. Let us suppose, therefore, that line ad cuts lines P_1b, P_1c, and P_1e in velocity-centers 12, 13, and 23, respectively; then, Theorem VI, Property B,

$$\omega_{21} = \frac{P_2V_{21}}{P_2\text{-}12}, \quad \ldots\ldots\ldots\ldots\ldots \quad (17)$$

$$\omega_{31} = \frac{P_3V_{31}}{P_3\text{-}13}, \quad \ldots\ldots\ldots\ldots\ldots \quad (18)$$

$$\omega_{23} = \frac{P_2V_{23}}{P_2\text{-}23}, \quad \ldots\ldots\ldots\ldots\ldots \quad (19)$$

in which ω_{21} of equation (17) is the angular velocity in radians per minute of body 2 relative to body 1 if P_2V_{21} is expressed in feet per minute and P_2-12 in feet. Similarly for the other equations.

Thus from the given triangle of velocities we have found the velocity-centers and the angular velocities of the three corresponding rigid bodies.

It should be observed that since line *ad* may be chosen arbitrarily in the plane of the velocity-triangle, it may be drawn so as to cut any one or all three of the lines P_1b, P_1c, P_1e at infinity. Thus we infer that the centers may be so chosen that the relative motion of any pair or of all three pairs of the rigid bodies may be *translation.*

We propose now in conclusion to give an application of Theorem XI to a simple mechanism. We choose for the purpose the 4-link linkage shown in Fig. 32. For this mechanism let us solve the following problem:

Problem 1.--For the 4-link linkage of Fig. 32 given; (1) the lengths of the links, (2) the angle θ, which fixes the configuration of the linkage, (3) N_{21}, the angular velocity in R.P.M. of crank 2 relative to fixed link 1, (4) the position of a point P_4 in lever 4; required: (1) the velocity of point P_4 relative to link 1, that is, P_4V_{41}, (2) the velocity of point P_4 relative to link 2, that is, P_4V_{42}.

Solution.--The rigid bodies or links involved in the required velocity-triangle are evidently 1, 2, and 4; hence find velocity-centers 12, 14, 24 for these three links. (These centers are, it is well known, Theorem VII, located as shown in the figure.) With point P_4 of link 4 coincide points P_1 and P_2 of links 1 and 2, respectively. Hence symbolize point P_4 as shown in the figure. Now join these three coincident points to the three velocity-centers 12, 14, 24; compute the value of P_2V_{21} from the relation

$$P_2V_{21} = 2\pi N_{21}(P_2\text{-}12); \quad \ldots\ldots\ldots\ldots\ldots \quad (20)$$

lay off P_1b at right angles to P_2-12 and, to some convenient velocity scale, equal to P_2V_{21}; through P_1 draw P_1c at right angles to P_1-14; and through b draw bc at right angles to P_2-24. Then, Theorem XI,

$$P_4c = P_4V_{41}, \quad \ldots\ldots\ldots\ldots\ldots \quad (21)$$

and

$$bc = P_4V_{42} = -P_2V_{24}, \quad \ldots\ldots\ldots\ldots\ldots \quad (22)$$

each to be measured, of course, to the velocity scale chosen for P_2V_{21}.

Let it be noted in connection with the solution of this problem that point P_4 may be taken anywhere in link 4 except on line 12-14. Let it be noted, moreover, that velocity-triangle P_1bc can be drawn without difficulty even when velocity-center 24 goes, as of course it does for many configuration of the mechanism, beyond the limits of the drawing.

Corollary I to Theorem IX.--If three rigid bodies are undergoing at and for a given instant complanar relative motion, the rotation of the first of these bodies relative to the second may be resolved into a rotation of the first, relative to the third and a rotation of the third relative to the second.

Proof.--Theorem IX has shown us that when we have three rigid bodies, 1, 2, 3, undergoing instantaneous complanar motion the three associated angular velocities are at each and every instant of the motion connected by the equation

$$N_{21} = N_{31} + N_{23}. \quad \ldots\ldots\ldots\ldots\ldots \quad (23)$$

That is to say, therefore, the angular velocity of body 2 relative to body 1 about axis 21 is equal to the algebraic sum of the angular velocity of body 2 relative to body 3 about axis 23 plus the angular velocity of body 3 relative to body 1 about axis 31. But, this is, evidently, just another way of saying that this theorem shows us that the angular velocity of body 2 relative to body 1 about axis 21 may be resolved into two other simultaneous rotations: (1) a rotation of body 2 relative to body 3 about axis 23, and (2) a rotation of body 3 relative to body 1 about axis 31. But this was what was to be proved.

Corollary II to Theorem IX.--From what has just been proved we conclude that if we have a rigid body 2 rotating at and for a given instant relative to another rigid body 1 about axis 12, this rotation can be resolved into two other simultaneous rotations in an infinite number of ways. Because through velocity-center 12 an infinite number of straight lines can be drawn in the plane of motion, and on any one of these lines center 13 (or 23) can be chosen in an infinity of ways, and for each chosen position of 13 (or 23) center 23 (or 13) can be chosen in an infinity of ways.

Corollary III to Theorem IX.--We conclude, moreover, that since either of these centers, 13 and 23, may be taken at infinity on the line that joins it to center 12, the rotation of body 2 relative to body 1 about axis 12 may be resolved into a rotation and a translation in an infinite number of ways. Because through velocity-center 12 an infinite number of lines can be drawn, and on any one of these lines with 13 (or 23) at infinity 23 (or 13) can be chosen in an infinite number of ways.

Caution.--The student should be very sure to observe here in considering these corollaries to Theorem IX that the rotation of body 2 relative to body 1 about axis 12 is not resolved into two other rotations of body 2 about two other axes relative to body 1, but, as the symbolization shows clearly, into two other rotations, of which one is a rotation of body 2 relative to body 3 about axis 23 and the other a rotation of body 3 relative to body 1 about axis 31.

Unfortunately, not many books on mechanics take the pains to make this matter clear; in reading such books, therefore, the student will be confused unless he keeps in mind the caution that has just been given him.

Theorem XII, *Rankine's Theorem of Longitudinal Components.*--If at and for a given instant body 2 is undergoing complanar motion relative to body 1, that is, rotating relative to body 1 about axis 21, and the velocities of any two points in body 2 be resolved along and at right angles to the line joining the two points, the components along this line are equal.

Let body 2, Fig. 33, be rotating relative to body 1 about velocity-axis 21, let P_2 and Q_2 designate *any* two points in body 2, and let P_2m and Q_2n, drawn to scale and at right angles to P_2-*12* and Q_2-*12*, respectively, represent the linear velocities of these two points relative to body 1; in other words, let P_2m and Q_2n represent P_2V_{21} and Q_2V_{21}, respectively. Resolve each of these velocities into two components, one along P_2Q_2, the other at right angles thereto. Then if P_2p and Q_2q are the components of these velocities along P_2Q_2, Rankine's theorem asserts that

$$P_2p = Q_2q. \quad \ldots\ldots\ldots\ldots\ldots \quad (24)$$

Rankine's proof of this proposition is very short and easily stated, and for these reasons appeals strongly to students, but despite the fact that Rankine was an intellectual engineering giant, his proof is faulty. Nevertheless, however, it is the proof that has been adopted by all writers on the science of mechanism since Rankine's day. Rankine's argument in support of the truth of his theorem is that equation (24) must be true because otherwise the distance P_2Q_2 would have to change; but that this distance cannot change, for the points are in the same rigid body; therefore, Rankine said, the proposition must be true. Unless, however, it should be remembered, we know about a linear velocity the five things stated in Postulate II, p. 43, we cannot deal with it intelligently. Now about no one of the four velocities P_2p, Q_2q, pm, and qn do we know all five of these things. Thus Rankine's treatment of the proposition must be considered unsatisfactory.

The principal objection to Rankine's way of looking at the question is that it does not lend itself to the explanation or interpretation of these four linear velocities. Our treatment of the theorem will not be subject to this objection.

Proof of Theorem XII.--When linear velocity $P_2m(=P_2V_{21})$ is resolved into the components P_2p and pm, a triangle of velocities--P_2mp--is thus formed and introduced into the question,

and along with it, of course, a third rigid body 3 with its two additional velocity-centers 13 and 23. See Corollary I, Theorem XI, p. 46. Not only so, but now that there are three rigid bodies in the system, with point P_2 in body 2 there must coincide two other points, P_1 in body 1 and P_3 in body 3. Similarly for point Q_2: with it must coincide points Q_1 and Q_3. Moreover, to satisfy the Kennedy-Aronhold theorem of three velocity-centers, Theorem VII, centers 12, 13, 23 must lie on the same straight line; that is, these two additional centers 13 and 23 must lie on some straight line passing through center 12. Moreover, Theorem XI, these three centers must be so located that (1) when joined to point P_2, the lines P_2-12, P_2-13, and P_2-23 will be perpendicular, respectively, to the sides of the triangle P_2pm, and (2) when joined to point Q_2 the lines Q_2-12, Q_2-13, Q_2-23 will likewise be perpendicular, respectively, to the sides of the triangle Q_2qn.

To satisfy all three of these conditions, the Kennedy-Aronhold condition as well as the two just stated, centers 13 and 23 must evidently be located as shown in Fig. 33; that is, (1) they must lie on the straight line that passes through velocity center 12 at right angles to P_2Q_2, (2) center 23 (or 13) must be the point of intersection of this line with P_2Q_2, and (3) center 13 (or 23) must be the point on this same line at infinity.

We are now in a position not only to prove the truth of the equation

$$P_2p = Q_2q. \quad \ldots\ldots\ldots\ldots\ldots \quad (24)$$

but to interpret any linear-velocity vector entering into the solution.

Keeping always in mind Theorem XI, the *Theorem of the Triangle of Velocities*, we see, first, that, since side P_2p of velocity triangle P_2pm is perpendicular to the line that joins P_3 to 13 (or 23), P_2p is either (1) the linear velocity of point P_3 (a point in body 3) relative to body 1, that is P_3V_{31}, or (2) the linear velocity of point P_2 (a point in body 2) relative to body 3, that is, P_2V_{23}, according as the center at infinity is 13 or 23. We see, secondly, that, since side $pm(=P_2p'_2)$ of velocity triangle P_2pm is perpendicular to the line that joins P_2 to 23 (or 13), pm is either (1) the linear velocity of point P_2 (a point in body 2) relative to body 3 or (2) the linear velocity of point P_3 (a point in body 3) relative to body 1, according as the foot of the perpendicular from velocity-center 12 to P_2Q_2 is 23 or 31. In an exactly similar manner we see: (1) that Q_2q is either Q_3V_{31} or Q_2V_{23} according as the center at infinity is 13 or 23; (2) that $qn(=Q_2q'_2)$ is either Q_2V_{23} or Q_3V_{31} according as the foot of the perpendicular from 12 to P_2Q_2 is 23 or 13.

Moreover, (1) if on the one hand, center 13 is the center at infinity body 3 has translatory motion relative to body 1, and thus the linear velocities of all points in body 3 relative to body 1 are equal; hence

$$P_3V_{31} = Q_3V_{31}, \quad \ldots\ldots\ldots\ldots\ldots \quad (25)$$

that is to say,

$$P_2p = Q_2q;$$

(2) if, on the other hand, center 23 is the center at infinity body 2 has translatory motion relative to body 3 and thus the linear velocities of all points in body 2 relative to body 3 are equal; hence

$$P_2V_{23} = Q_2V_{23}, \quad \ldots\ldots\ldots\ldots\ldots \quad (26)$$

or, in other words,

$$P_2p = Q_2q. \quad \ldots\ldots\ldots\ldots\ldots \quad (24'')$$

In either case therefore, whether center 13 or center 23 is the center at infinity, Rankine's theorem is seen to be true.

We have now proved Rankine's theorem, but there is a great deal more to it than what we have put down. Let us look at it from another angle.

As we have seen, the proof of Rankine's theorem requires a system of three rigid bodies; in other words, with only two bodies in the system it is impossible to make a scientific discussion of the proposition, because then no adequate interpretation of any one of the vectors P_2p, Q_2q, pm, qn, can be given.

With three bodies in the system, however, we have, Theorem IX, the following relation connecting the three angular velocities associated with this system of bodies:

$$N_{21} = N_{31} + N_{23},$$

that is,

$$2\pi N_{21} = 2\pi N_{31} + 2\pi N_{23},$$

or

$$\omega_{21} = \omega_{31} + \omega_{23}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

But either ω_{31} or ω_{23} is zero, according as center 31 or 23 is the center at infinity on the line through center 12 perpendicular to P_2Q_2. Therefore (1) $\omega_{21} = \omega_{23}$ or (2) $\omega_{21} = \omega_{31}$, according as center 31 or 23 is the center at infinity. There are, therefore two cases according as center 13 or 23 is at infinity.

First Case, Center 13 at Infinity.--In this case $\omega_{21} = \omega_{23}$, that is, body 2 at the instant is turning around axis 23 relative to body 3 at the same rate and in the same sense that it is turning around axis 12 relative to body 1.

Therefore, Theorem VI, Property B,

$$\omega_{21} = \frac{P_2V_{21}}{P_2\text{-}21} = \frac{Q_2V_{21}}{Q_2\text{-}21}$$

$$= \frac{P_2m}{P_2\text{-}21} = \frac{Q_2n}{Q_2\text{-}21}; \quad \ldots\ldots\ldots\ldots \quad (28)$$

but

$$\frac{P_2m}{P_2\text{-}21} = \tan \alpha_p, \quad \ldots\ldots\ldots\ldots \quad (29)$$

and

$$\frac{Q_2n}{Q_2\text{-}21} = \tan \alpha_q; \quad \ldots\ldots\ldots\ldots \quad (30)$$

therefore, since α_p and α_q must (why?) always be less than 90°,

$$\alpha_p = \alpha_q. \quad \ldots\ldots\ldots\ldots \quad (31)$$

In exactly the same way we have also

$$\omega_{23} = \frac{P_2V_{23}}{P_2\text{-}23} = \frac{Q_2V_{23}}{Q_2\text{-}23}$$

$$= \frac{P_2p'_2}{P_2\text{-}23} = \frac{Q_2q'_2}{Q_2\text{-}23}; \quad \ldots\ldots\ldots\ldots \quad (32)$$

but

$$\frac{P_2 p_2'}{P_2\text{-}23} = \tan \beta_p , \quad \ldots\ldots\ldots\ldots\ldots \quad (33)$$

and

$$\frac{Q_2 q_2'}{Q_2\text{-}23} = \tan \gamma_q ; \quad \ldots\ldots\ldots\ldots\ldots \quad (34)$$

therefore, since β_p and γ_q must always be less than 90°,

$$\beta_p = \gamma_q . \quad \ldots\ldots\ldots\ldots\ldots \quad (35)$$

Moreover, for this case, since 13 is at infinity and therefore ω_{31} in equation (27) is zero,

$$\omega_{21} = \omega_{23} , \quad \ldots\ldots\ldots\ldots\ldots \quad (36)$$

that is, equations (28) and (32),

$$\frac{P_2 m}{P_2\text{-}21} = \frac{P_2 p_2'}{P_2\text{-}23} , \quad \ldots\ldots\ldots\ldots\ldots \quad (37)$$

and hence

$$\alpha_p = \alpha_q = \beta_p = \gamma_q . \quad \ldots\ldots\ldots\ldots\ldots \quad (38)$$

Since angles β_p and γ_q of triangles $P_2 p_2' (23)$ and $Q_2 q_2' (23)$ are equal and sides $P_2\text{-}23$ and $Q_2\text{-}23$ collinear, vertices p_2', 23, and q_2' are also collinear; that is to say, the line joining the ends of any two side components of the velocities of any two points in body 2 passes through velocity-center 23.

Second Case, Center 23 at Infinity.--This case, in which velocity-center 23 is at infinity and velocity-center 31 at the foot of the perpendicular from velocity-center 12 to line $P_2 Q_2$, if handled in the same way as the first case, yields the same conclusions.

The student should be sure, nevertheless, to go through the argument for the First Case and everywhere in it interchange centers 23 and 13 and see that thus a valied argument for the Second Case is derived.

Construction for Finding the Velocity of any Other Point in Line $P_2 Q_2$--1. The velocity of any other point R_2 of body 2 on line $P_2 Q_2$ (either between P_2 and Q_2 or not) can, in consequence of what has just been proved, be found as follows when center 12 and linear velocity $P_2 V_{21}$ are known: (1) Through center 12 draw 12-13 perpendicular to $P_2 Q_2$, thus determining center 23; (2) resolve $P_2 V_{21}$ along and at right angles to $P_2 Q_2$, thus finding the components $P_2 p$ and mp, the latter being equal and parallel to $P_2 p_2'$; (3) lay off $R_2 o$ from R_2 equal to $P_2 p$ (why?); (4) join 23 to p_2', (5) through R_2 draw $R_2 r_2'$ perpendicular to $R_2\text{-}23$ and produce it to cut $23\text{-}p_2'$ at r_2'; (6) make or equal and parallel to $R_2 r_2'$; (7) join R_2 to r. Then $R_2 r$ represents $P_2 V_{21}$ to the same scale that $P_2 m$ represents $P_2 V_{21}$. Or point r, the end of the vector representing the velocity of point R_2 relative to body 1, may be found as follows: (1) Through R_2 draw $R_2 r$ perpendicular to $12\text{-}R_2$; (2) from R_2 along $P_2 Q_2$ lay off $R_2 o$ equal to $P_2 p$; (3) at o set up the perpendicular to $P_2 Q_2$; this perpendicular will cut $R_2 r$ at r.

2. There is another and a somewhat more convenient way to find vector $R_2 r$, the vector that represents $P_2 V_{21}$. But this method requires that the velocities of two points collinear with R_2 in body 2 relative to body 1 be known. Suppose we know $P_2 V_{21}$ and $Q_2 V_{21}$. Then:

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(1) join points m and n, the extremities of the vectors that represent $_{P_2}V_{21}$ and $_{Q_2}V_{21}$; (2) lay off R_2o equal in magnitude and sense to P_2p; (3) at o draw or perpendicular to P_2Q_2 and produce it to cut mn in r; r is then, as Theorem XII_1 below shows, the end of vector R_2r, the vector that represents $_{R_2}V_{21}$.

Proof.--The proof depends upon a simple and well-known kinematic principle, which may be stated as follows: If P_2, Q_2, S_2 ... are points in a rigid body 2, Fig.33a, which is rotating at and for a given instant relative to a second body 1 about axis 12, and if vectors P_2P_2', Q_2Q_2', S_2S_2' ... represent the linear velocities of these points, then the polygon whose vertices are P_2', Q_2', S_2' ... is similar to the polygon whose vertices are P_2, Q_2, S_2 ... .

To prove this proposition we make use several times of the well-known geometrical theorem--the XVIIIth Proposition of the Third Book of Euclid--that if (1) an angle of one triangle is equal to an angle of a second triangle and (2) the corresponding sides of the two triangles about these equal angles are in proportion, the triangles are similar.

(1) From Theorem VI, Property B, we have, Fig. 33a,

$$\frac{_{P_2}V_{21}}{_{Q_2}V_{21}} = \frac{P_2P_2'}{Q_2Q_2'} = \frac{2\pi N_{21}(P_2-21)}{2\pi N_{21}(Q_2-21)}, \quad \ldots\ldots\ldots\ldots \quad (39)$$

that is,

$$\frac{P_2P_2'}{Q_2Q_2'} = \frac{P_2-21}{Q_2-21}. \quad \ldots\ldots\ldots\ldots \quad (40)$$

Moreover, angles $(12)P_2P_2'$ and $(12)Q_2Q_2'$ of triangles (12), P_2, P_2' and (12), Q_2, Q_2' are right angles and therefore equal. Thus it follows from Euclid's theorem that these two triangles are similar. But if triangles (12), P_2, P_2' and (12), Q_2, Q_2' are similar

$$\angle P_2(12)Q_2 = \angle P_2'(12)Q', \quad \ldots\ldots\ldots\ldots \quad (41)$$

and also

$$\frac{P_2-12}{P_2'-12} = \frac{Q_2-12}{Q_2'-12}; \quad \ldots\ldots\ldots\ldots \quad (42)$$

thus, according to Euclid's theorem, triangles (12), P_2, Q_2 and (12), P_2', Q_2' are similar.

(2) By exactly the same reasoning it can be proved that triangles (12), P_2, S_2 and (12), P_2', S_2' are similar. But if triangle (12), P_2, Q_2 is similar to triangle (12), P_2', Q_2' and triangle (12), P_2, S_2, is similar to triangle (12), P_2', S_2', then

$$\angle Q_2P_2S_2 = \angle Q_2'P_2'S_2'; \quad \ldots\ldots\ldots\ldots \quad (43)$$

moreover, from similar triangles (12), P_2, Q_2 and (12), P_2', Q_2'

$$\frac{P_2Q_2}{P_2'Q_2'} = \frac{P_2-12}{P_2'-12}, \quad \ldots\ldots\ldots\ldots \quad (44)$$

and, from similar triangles (12), P_2, S_2 and (12), P_2', S_2'

$$\frac{P_2S_2}{P_2'S_2'} = \frac{P_2-12}{P_2'-12}; \quad \ldots\ldots\ldots\ldots \quad (45)$$

therefore, equations (44) and (45),

$$\frac{P_2Q_2}{P_2'Q_2'} = \frac{P_2S_2}{P_2'S_2'}; \quad \ldots\ldots\ldots\ldots \quad (46)$$

and thus it follows from equations (43) and (46), by Euclid's theorem, that triangles P_2, Q_2, S_2 and P_2', Q_2', S_2' are similar.

(3) But if the proposition in question is true for a polygon of *three* vertices it is obviously true for a polygon of any number of vertices, since any polygon is composed of triangles. Thus the proposition is true in general.

(4) A special case of the proposition, the case with which at the moment we are mainly concerned, arises when points P_2, Q_2, S_2 ... are collinear. For this case, obviously, P_2', Q_2', S_2' ... are also collinear. Hence we have the useful kinematic proposition:

Theorem XII_1.--If a rigid body 2 is undergoing at and for an instant complanar motion relative to a second rigid body 1, and if P_2, Q_2, S_2 ... designate any number of *collinear* points belonging to body 2, then P_2', Q_2', S_2' ..., the ends of the vectors representing the linear velocities of points P_2, Q_2, S_2 ..., are also collinear.

We begin now the discussion of an entirely different and the most elegant, and usually the best, way to go from the linear velocity of any point in one link in a mechanism to the linear velocity of any point in any other link of the mechanism. This is the method of velocity-centers. It makes use of the principle enunciated in Theorem VI, Property A.

Theorem XIII, *The Kennedy Theorem of Two Linear Velocities in Two Rigid Bodies Undergoing Complanar Motion Relative to a Third Rigid Body.*--If P_2 and P_3 designate points anywhere in rigid bodies 2 and 3, respectively, and if these bodies are undergoing complanar motion relative to a third rigid body 1, then the relation connecting the linear velocities of points P_2 and P_3 relative to body 1 is

$$\frac{P_3V_{31}}{Q_2V_{21}} = \frac{P_3\text{-}31}{P_2\text{-}21} \times \frac{23\text{-}21}{13\text{-}23}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (47)$$

Proof.--Since points P_2 and 23, Fig. 34, are points in body 2 at the instant under consideration, Theorem VI, Property B, gives us the relation

$$\frac{_{23}V_{21}}{P_2V_{21}} = \frac{23\text{-}21}{P_2\text{-}21}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (48)$$

and since, likewise, points P_3 and 23 are points in body 3 at the same instant, we have, similarly,

$$\frac{P_3V_{31}}{_{23}V_{31}} = \frac{P_3\text{-}31}{23\text{-}31}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (49)$$

therefore, combining equations (48) and (49), and remembering that $_{23}V_{21} = {}_{23}V_{31}$

$$\frac{P_3V_{31}}{P_2V_{21}} = \frac{P_3\text{-}31}{P_2\text{-}21} \times \frac{23\text{-}21}{23\text{-}31}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (50)$$

To find P_3V_{31} when P_2V_{21} is given we must know, equation (47) tells us: (1) the relative locations of the three velocity-centers 12, 23, and 13; (2) the distance of point P_2 from center 12, and (3) the distance of P_3 from center 13.

Graphical Solution of Equation (47).--Let points P_2 and P_3 and the three centers be located as shown in Fig. 34. Let P_2p_2, perpendicular to $P_2\text{-}12$, represent P_2V_{21} to scale; make right-angle triangle 12, 23, r similar to right-angle triangle 12, P_2, p_2; make right-angle triangle 13, P_3, p_3 similar to right-angle triangle 13, 23, r. Then because right-angle triangle 12, 23, r is similar to right-angle triangle 12, P_2, p_2 side $23\text{-}r$ represents $_{23}V_{21} (= {}_{23}V_{31})$ to the same scale to which P_2p_2 represents P_2V_{21}, and we have

$$\frac{P_2V_{21}}{_{23}V_{21}} = \frac{P_2\text{-}21}{23\text{-}21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (51)$$

Similarly, because right-angle triangle 13, P_3, p_3 is similar to right-angle triangle 13, 23, r, side P_3p_3 of the former represents ${}_3V_{31}$ to the same scale to which corresponding side $23\text{-}r$ of triangle 13, 23, r represents ${}_{23}V_{31}$, and we have

$$\frac{P_3V_{31}}{{}_{23}V_{31}} = \frac{P_3\text{-}31}{23\text{-}31}. \quad \ldots\ldots\ldots\ldots\ldots \quad (52)$$

From equations (51) and (52), eliminating ${}_{23}V_{21}(={}_{23}V_{31})$, we get

$$\frac{P_3V_{31}}{P_2V_{21}} = \frac{12\text{-}23}{P_2\text{-}21} \times \frac{P_3\text{-}31}{23\text{-}31},$$

which is the same as equation (47).

Comments.--It will be noticed that this method of determining linear velocities requires the use of three velocity-centers: (1) the velocity-center of the body of reference and the body that carries the point whose linear velocity is given, that is, velocity-center 21; (2) the velocity-center of the body of reference and the body that carries the point whose linear velocity is required, that is, velocity-center 13; (3) the velocity-center of the bodies that carry the two given points, that is, velocity-center 23. And whenever in any case these three centers are available this is the easiest, the most elegant, and the best way to solve the problem with which we are dealing. For certain configurations of nearly all mechanisms, however, at least one of these three centers goes beyond the limits of the paper, and when this happens this method of velocity-centers fails to give a solution, and then some of the other methods must be used. Moreover, for any except the simpler mechanisms it is often troublesome to find one or more of the necessary centers.

One other method of finding linear velocities in mechanisms remains to be explained, namely, the very excellent and elegant method of velocity-images or the method of the polar-velocity diagram. To the explanation of this method we now proceed.

Theorem XIV, *Burmester's Theorem of Velocity Images.*--If P_2, Q_2, S_2 ... designate points in a rigid body 2, which at and for a given instant is undergoing complanar motion relative to a body 1, and if vectors (all to the same velocity scale) be drawn from a pole to represent in magnitude, direction, and sense the velocities of these several points relative to body 1, then the ends of these velocity vectors will be the vertices of a polygon similar to the polygon whose vertices are P_2, Q_2, S_2 ... .

We shall usually refer to this method as the method of velocity images. Of all the methods so far devised for determining linear velocities in mechanisms this method, all things considered, is unquestionably the best. It is therefore deserving of the closest attention of the student of mechanism.

Proof of the Theorem.--In Fig. 35a two rigid bodies that are undergoing complanar relative motion are shown; that is, body 2 is revolving relative to body 1 about axis 12 (which may be either instantaneous or permanent) at a known rate and in a known sense. The rate is N_{21} revolutions per minute and the sense is clockwise, let us suppose. P_2 and Q_2 are any two points in body 2 at known distances from velocity center 12. Then Theorem VI, Property B,

$$P_2V_{21} = 2\pi N_{21}(P_2\text{-}21), \quad \ldots\ldots\ldots\ldots\ldots \quad (53)$$

and

$$Q_2V_{21} = 2\pi N_{21}(Q_2\text{-}21). \quad \ldots\ldots\ldots\ldots\ldots \quad (54)$$

In Fig. 35b draw $12'\text{-}P_2'$ perpendicular to $12\text{-}P_2$ to represent P_2V_{21} to any convenient velocity scale. Similarly, draw $12'\text{-}Q_2'$ perpendicular to $12\text{-}Q_2$, and to the same scale, to represent Q_2V_{21}. Join P_2' and Q_2'.

Then triangles 12, P_2, Q_2 and 12', P'_2, Q'_2 are similar, because (1) angle α of the first is equal to angle α' of the second and (2) the corresponding sides about these two equal angles are in proportion. Angle α is equal to angle α' because $12\text{-}P_2$ is perpendicular to $12'\text{-}P'_2$ and $12\text{-}Q_2$ to $12'\text{-}Q'_2$. The corresponding sides about the equal angles are in proportion, because, Theorem VI, Property B,

$$\frac{P_2V_{21}}{Q_2V_{21}} = \frac{2\pi N_{21}(12\text{-}P_2)}{2\pi N_{21}(12\text{-}Q_2)} = \frac{12\text{-}P_2}{12\text{-}Q_2}; \quad \ldots\ldots\ldots\ldots \quad (55)$$

but

$$P_2V_{21} = 12'\text{-}P'_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (56)$$

and

$$Q_2V_{21} = 12'\text{-}Q_2; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (57)$$

therefore

$$\frac{12\text{-}P_2}{12\text{-}Q_2} = \frac{12'\text{-}P'_2}{12'\text{-}Q'_2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (58)$$

Since, now triangles 12, P_2, Q_2 and 12', P'_2, Q'_2 are similar and two sides of the one are perpendicular respectively to two sides of the other, the third sides of these two triangles are also perpendicular. It follows therefore that the length of vector $12'\text{-}Q'_2$ could have been found as follows: (1) Through pole $12'$, Fig. 35b, draw a straight line of indefinite length perpendicular to $12\text{-}Q_2$ of Fig. 35a; (2) through point P'_2 draw an indefinite straight line perpendicular to line P_2Q_2 of Fig. 35a to meet the indefinite straight line that is perpendicular to $12\text{-}Q_2$ at point Q'_2 of Fig. 35b.

Let us take any third point S_2 in body 2 and find the vector in the velocity diagram that represents its velocity. Obviously, to do this we must deal with points P_2 and S_2 just as we dealt with P_2 and Q_2; that is, we draw through pole $12'$ an indefinite straight line perpendicular to $12\text{-}S_2$ and through P'_2 an indefinite straight line perpendicular to P_2S_2, thus finding $12'\text{-}S'_2$, the vector that represents the velocity of point S_2. It will be understood, of course, that vector $12'\text{-}S'_2$ could have been found by dealing with points Q_2 and S_2 instead of with points P_2 and S_2. Thus it follows that line $Q'_2S'_2$, Fig. 35b, is perpendicular to line Q_2S_2, Fig. 35a.

The student should be sure to notice that for the same reasons that the sides of triangles 12, P_2, Q_2 and $12'$, P'_2, Q'_2 are respectively perpendicular, it follows that the sides of triangles 12, Q_2, S_2 and $12'$, Q'_2, S'_2 are respectively perpendicular, and so also the respective sides of triangles 12, P_2, S_2 and $12'$, P'_2, S'_2. And that thus it follows that the sides of triangles $P_2Q_2S_2$ and $P'_2Q'_2S'_2$ are respectively perpendicular and the triangles similar.

Points $12'$, P'_2, Q'_2, S'_2, Fig. 35b, are called the velocity images of points 12, P_2, Q_2, S_2, Fig. 35a. Similarly, lines $12'\text{-}P'_2$, $12'\text{-}Q'_2$, $12'\text{-}S'_2$, $P'_2Q'_2$, $P'_2S'_2$, $Q'_2S'_2$, Fig. 35b, are the velocity images of lines $12\text{-}P_2$, $12\text{-}Q_2$, $12\text{-}S_2$, P_2Q_2, P_2S_2, Q_2S_2, Fig. 35a. And in the same way, triangles $(12')P'_2Q'_2$, $(12')P'_2S'_2$, $(12')Q'_2S'_2$, $P'_2Q'_2S'_2$, are the velocity images of triangles $(12)P_2Q_2$, $(12)P_2S_2$, $(12)Q_2S_2$, $P_2Q_2S_2$, Fig. 35a. Thus a line and its velocity image are perpendicular to each other and the sides of a triangle and the corresponding sides of its velocity image are perpendicular to each other.

There is another way to establish the relation between the mechanism diagram, Fig. 35a, and the polar-velocity diagram, Fig. 35b, that some students find easier to understand.

Draw triangle $12'$, P'_2, Q'_2, Fig. 35b, of any convenient size with its sides respectively perpendicular to the sides of triangle 12, P_2, Q_2. Then the two triangles are similar, and therefore

$$\frac{12'\text{-}P_2'}{12'\text{-}Q_2'} = \frac{12\text{-}P_2}{12\text{-}Q_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (59)$$

But, Theorem VI, Property B,

$$\frac{P_2V_{21}}{Q_2V_{21}} = \frac{2\pi N_{21}(12\text{-}P_2)}{2\ N_{21}(12\text{-}Q_2)} = \frac{12\text{-}P_2}{12\text{-}Q_2}; \quad \ldots\ldots\ldots\ldots\ldots \quad (60)$$

therefore

$$\frac{P_2V_{21}}{Q_2V_{21}} = \frac{12'\text{-}P_2'}{12'\text{-}Q_2'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (61)$$

Thus to whatever scale vector $12'\text{-}P_2'$ represents P_2V_{21}, vector $12'\text{-}Q_2'$ to the same scale represents Q_2V_{21}.

In exactly the same way if triangle $12'$, P_2', S_2' is drawn to any convenient size with its sides respectively perpendicular to the sides of triangle 12, P_2, S_2, then to whatever scale vector $12'\text{-}P_2'$ represents P_2V_{21} vector $12'\text{-}S_2'$ represents S_2V_{21} to the same scale

Interpretation of the Non-polar Vectors of the Velocity-Image Diagrams.--It is of the utmost importance that the student should know the meaning of each of the six velocity vectors in Fig. 35b. The meaning of the three vectors passing through the pole $12'$ is perfectly obvious: they are, as indicated in the figure, P_2V_{21}, Q_2V_{21}, S_2V_{21}, that is to say, they are the linear velocities of the three points P_2, Q_2, and S_2 of body 2 relative to body 1. But the meaning of each of the other three velocity vectors of Fig. 35b is not so obvious, indeed not obvious at all. Let us begin their explanation by interpreting vector $P_2'Q_2'$, Fig. 35b.

Unless, as will appear later on in the discussion, a third body is introduced into the system none of these three non-polar vectors can be scientifically interpreted, because, as we saw when studying the *Theorem of the Triangle of Velocities*, a velocity triangle--triangle $12'$, P_2', Q_2', Fig. 35b, in this case--always implied the presence of three rigid bodies in the system. We know, moreover, from our study of the *Theorem of the Triangle of Velocities* that this third body--we will call it 3--can be introduced in an infinite number of ways, subject of course to the condition that velocity-centers *12*, *13*, and *23*, must lie on the same straight line. These three centers may then be chosen as shown in Fig. 36a, which is exactly the same as Fig. 35a with two additional lines and two additional symbols added.

Line P_2C, one of the additional lines of Fig. 36a, is drawn through point P_2 parallel to line $Q_2\text{-}12$, and line $12\text{-}B$, the other additional line, is any straight line drawn through velocity-center *12*, cutting lines P_2C and P_2Q_2 in velocity-centers *13* and *23*, respectively. Now since we have three bodies in the system it is evidently permissible to consider point P_2 three coincident points--P_1 in body 1, P_2 in body 2, and P_3 in body 3. It is evident, moreover, that the sides of triangle $12'$, P_2', Q_2', Fig. 36b, are respectively perpendicular to the lines that join the three coincident points $P_1P_2P_3$ of Fig. 36a to velocity-centers *12*, *13*, and *23*. Thus, in accordance with Theorem XI, the *Theorem of the Triangle of Velocities*, sides $12'\text{-}P_2'$, $12'\text{-}Q_2'$, $P_2'Q_2'$ of velocity triangle $12'$, P_2', Q_2'--perpendicular respectively to the lines that join $P_1P_2P_3$ to velocity-centers *12*, *13*, and *23*--represent, respectively, the linear velocities P_2V_{21}, P_3V_{31}, and P_2V_{23}. In other words, we see that vector $Q_2'P_2'$, the side of triangle $12'$, P_2', Q_2' that does not pass through the pole, is the linear velocity of point P_2 in body 2 around velocity-center *23*, relative to body 3, the extra body that we have found it necessary to introduce into the system in order to get a scientific interpretation of vector $Q_2'P_2'$.

Line $12\text{-}B$, Fig. 36a, may, however, as we have seen, be drawn in any direction through center *12*; it may then be taken to coincide with line $12\text{-}Q_2$; in which case, Fig. 37a, center *13* will recede to infinity along line P_2C and center *23* will coincide with point Q_2. Hence in this case vector $Q_2'P_2'$ is the linear velocity of point P_2 around point Q_2 relative

to body 3. This interpretation of vector $Q_2'P_2'$ is true only, however, it should be noted, if its sense is from Q_2' to P_2'. If we take its sense as being from P_2' to Q_2' then $P_2'Q_2'$ is the velocity of point P_2 around point Q_2 relative to a second and a different extra body 3.

To show this, consider Fig. 38a, where line Q_2C is drawn through point Q_2 parallel to line *12*-P_2 and points Q_1 and Q_3 of bodies 1 and 3, respectively, are taken coincident with point Q_2. Through center *12* draw line *12*-*B* to cut Q_2C and P_2Q_2 in centers *13* and *23*, respectively. Then it will be seen that the sides of velocity-triangle *12'*, P_2', Q_2' are perpendicular respectively to the three lines that join the three coincident points Q_1, Q_2, Q_3 to the three velocity-centers *12*, *13*, *23*, and that thus side $P_2'Q_2'$, being perpendicular to *23*-Q_2, is the linear velocity of Q_2 around center *23* relative to body 3.

Now since transversal *12*-*B*, Fig. 38a, may be drawn in any direction through center *12*, it may be taken to coincide with line *12*-P_2, as shown in Fig. 39a. Then center *23* will coincide with point P_2 and center *13* will recede to infinity on line Q_2C. Hence, as asserted vector $P_2'Q_2'$ is the linear velocity of point Q_2 around point P_2 relative to body 3,

The student is apt to be puzzled about the *sense* of the vector that represents the velocity of point P_2 around Q_2 or the velocity of point Q_2 around P_2. There is, however, really, no difficulty about the matter, as the following explanation shows.

The question is: in which sense must side *D* of velocity triangle *12'*, P_2', Q_2', Fig. 37b, be read that it may be the velocity of point P_2 around Q_2 or *23* relative to body 3? There are two ways of settling the question.

First Method of Determining the Sense of Vector D.--Since there are three bodies in the system undergoing complanar motion, we have, Theorem IX,

$$N_{21} = N_{31} + N_{23}.$$

But N_{31} is zero because velocity-center *31* is by hypothesis at infinity. Therefore

$$N_{21} = N_{23}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (62)$$

in other words, body 2 is turning about velocity-center *23* relative to body 3 in the same sense as that in which it is turning about center *12* relative to body 1. But the sense of vector *12'*-P_2', Fig. 37b, shows that body 2 is turning clockwise about center *12* relative to body 1; thus, equation (62), it follows that body 2 is also turning clockwise about center *23* relative to body 3. Now if body 2 is turning clockwise about center *23* relative to body 3, point P_2 of body 2 must be moving in the sense from Q_2' to P_2' relative to body 3.

Second Method of Determining the Sense of Vector D.--Consider velocity-triangle *12'*, P_2', Q_2', Fig. 37b. The sides of this triangle are, as has already been pointed out, perpendicular, respectively, to the lines that join coincident points $P_1P_2P_3$ to velocity-centers *12*, *13*, *23*. Thus it follows: (1) that side *12'*-P_2'--perpendicular to *12*-P_2--read in one sense represents P_2V_{21}; read in the opposite sense it represents P_1V_{12}; (2) that side *12'*-Q_2'--perpendicular to *12*-Q_2--read in one sense represents Q_2V_{21}; read in the opposite sense it represents Q_1V_{12}; (3) that side *D*--perpendicular to P_2Q_2--read in one sense represents P_2V_{23}; read in the opposite sense it represents Q_2V_{23}. From velocity triangle *12'*, P_2', Q_2' we have, moreover, by Theorem XI the vector equation

$$P_2V_{21} = P_3V_{31} \leftrightarrow P_2V_{23} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (63)$$

concerning which we know that the sides of the triangle represented by the two terms on the right-hand side of the expression must be read around the triangle in the sense opposite to the sense in which the side of the triangle represented by the term on the left-hand side of the equation is read. But by hypothesis the side of the triangle represented by the left-hand side of the equation is read clockwise around the triangle; hence the other two sides must be read in the counterclockwise sense around the triangle. Thus to correctly represent the sense of P_2V_{23}, and not the sense of Q_2V_{23}, vector *D* must be read

from Q_2' towards P_2'--the same result as that obtained by the First Method.

Similarly, by Theorem XI again, we have for Fig. 39b, in which center *23* coincides with point P_2 and center *13* is at infinity on *12*-P_2.

$$Q_2V_{21} = Q_3V_{31} \nrightarrow Q_2V_{23}. \quad \ldots\ldots\ldots\ldots\ldots \quad (63')$$

From which, since Q_2V_{21} points around triangle *12'*, P_2', Q_2' counterclockwise, we see by the Rule of Senses that the sense of Q_2V_{23} is from P_2' towards Q_2'.

To get the sense of vector *D*, then, we have the simple rule:

> *Rule of Senses.*--The sense of vector *D* from Q_2' to P_2' is the sense of the velocity of point P_2 about point Q_2 relative to extra body 3; the sense from P_2' to Q_2', on the other hand, is the sense of the velocity of point Q_2 about point P_2 relative to a second and a different extra body 3.

Caution.--The student should be certain to notice that although the extra body of reference is designated with the numeral 3 in both cases, that is, whether we are fixing the sense of P_2V_{23} or the sense of Q_2V_{23}, these two extra bodies of reference are not one and the same but very different bodies, because in the first case center *23* coincides with Q_2 and center *13* is at infinity on line *12*-Q_2, while in the second case center *23* coincides with P_2 and center *13* is at infinity on line *12*-P_2.

Let us now return to Fig. 35b and in the light of what has just been said interpret the three non-polar vectors *D*, *E*, *F* of that figure.

Interpretation of Non-Polar Vector D.--(a) This vector read in the sense $Q_2'P_2'$ is P_2V_{23}. For this case the extra body is 3, center *13* is at infinity on line *12*-Q_2, and center *23* coincides with point Q_2. (b) Thus vector *D* read in the sense $P_2'Q_2'$ is Q_2V_{23}. For this case the extra body is 3, but now center *13* is at infinity on line *12*-P_2 and center *23* coincides with point P_2.

Interpretation of Non-Polar Vector E.--(a) This vector read in the sense $Q_2'S_2'$ is S_2V_{24}. For this case the extra body must be a fourth body, body 4, because now we are dealing with a new velocity-triangle *12'*, Q_2', S_2'. Here, moreover, center *14* is at infinity on line *12*-Q_2 and center *24* coincides with point Q_2. (b) Vector *E* read in the sense $S_2'Q_2'$ is Q_2V_{24}. For this case the extra body is also 4, center *14* is at infinity on line *12*-S_2, and center *24* coincides with point S_2.

Interpretation of Non-Polar Vector F.--(a) Read in the sense $P_2'S_2'$, this vector is S_2V_{25}. A different extra body, body 5, is necessary. Velocity-center *15* for this case is at infinity on line *12*-P_2 and velocity-center *25* coincides with P_2. (b) Read in the sense $S_2'P_2'$ vector *F* is P_2V_{25}, the extra body is 5, center *15* is at infinity on line *12*-S_2, and center *25* coincides with point S_2.

III. Analysis of Gear Trains

Solution of Problems X-13, X-14, X-15, X-16, X-17, X-25, X-32, X-34, X-39; XII-10, XII-13, XII-16, XII-17, XII-18

Problem X-25.--*Composition of the Train.* This problem concerns a 7-link gear-train of two baselinks, Fig. X-25, these notes. The train is composed of *n-2(=5)* cubic (3-link) spur-gear trains as follows: 145, 156, 167, 123, 234, We may also think of it as being composed of the general unibasal quintic 14567 and the reverted unibasal quartic 1234, these two trains having links 1 and 4 in common.

How the Mechanism Works. If we think of baselink 2 as being stationary and of duplex gear 3 as turning as a driver clockwise about axis 23 relative to link 2, gear 1 will be forced to turn counterclockwise about axis 12 and annular gear 4 clockwise about axis 24. But gear 1 is rigidly fastened to baselink 1 of unibasal quintic 14567 and annular gear 4 is rigidly fastened to gear 4 of this quintic, so that baselink 1 and gear 4 will be forced to turn in opposite senses about coincident axes 12 and 14 at speeds that will bear a certain constant ratio to each other. What this ratio of speeds is easily appears from the two basic ratios of the reverted unibasal quartic, 1234. These ratios are

$$\frac{N_{12}}{N_{32}} = -\frac{Z_3}{Z_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{N_{32}}{N_{42}} = \frac{Z'_4}{Z'_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

From which, multiplying the first by the second,

$$\frac{N_{12}}{N_{42}} = \frac{N_{12}}{N_{32}} \times \frac{N_{32}}{N_{42}} = -\frac{Z_3}{Z_1} \times \frac{Z'_4}{Z_3} \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

And if we choose

$$Z_1 = 20,\ Z_3 = 20,\ Z'_3 = 15,\ Z'_4 = 25,$$

$$\frac{N_{12}}{N_{42}} = -\frac{20}{20} \times \frac{25}{15} = -\frac{5}{3} \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

that is to say, with these numbers of teeth on the four gears of the reverted unibasal quartic 1234, baselink 1 will make 5 turns counterclockwise relative to baselink 2 for every three clockwise turns of duplex gear 4 relative to baselink 2, and this is the value of N_{12}/N_{42} that the statement in the text book prescribes.

Referring to Fig. X-25, these notes, the problem proposed in the text book, p. 260, can, it is obvious, be stated very concisely and correctly as follows: If N_{42} and N_{12} are 300 and -500 rpm., respectively, what are the values and signs of N_{52}, N_{62}, and N_{72}?

Now, equation (1) above, if $N_{12} = -500$ and $Z_1 = Z_3 = 20$, N_{32} must be $+500$; hence we can solve the problem as it is here stated by finding expressions in terms of the Zs for the three cubic angular velocity ratios

$$\frac{N_{32}}{N_{52}},\ \frac{N_{32}}{N_{62}},\ \frac{N_{32}}{N_{72}}.$$

Derivation of Angular Velocity Ratios N_{32}/N_{52}, N_{32}/N_{62}, N_{32}/N_{72}.--First, we write down the *n-2(=5)* basic ratios for the mechanism. Three of these ratios come from the unibasal quintic 14567 and are

Basic Ratios – Group I

$$\frac{N_{41}}{N_{51}} = -\frac{Z_5}{Z_4} \quad . \quad . \quad . \quad (1); \qquad \frac{N_{51}}{N_{61}} = -\frac{Z_6}{Z_5'} \quad . \quad . \quad . \quad (2); \qquad \frac{N_{61}}{N_{71}} = -\frac{Z_7}{Z_6} \quad . \quad . \quad . \quad (3)$$

the other two come from the reverted unibasal quartic 1234, and are

Basic Ratios – Group II

$$\frac{N_{12}}{N_{32}} = -\frac{Z_3}{Z_1} \quad . \quad . \quad . \quad (4), \qquad \frac{N_{32}}{N_{42}} = \frac{Z_4'}{Z_3'} \quad . \quad . \quad . \quad (5)$$

It will be noticed from the symbolism that baselink 1 is the link of reference for the three ratios of Group I, and baselink 2 the link of reference for those of Group II.

Our first objective now is to find a linkgroup of ratios in one of the n (=7) rows of the table for septemic trains, p. 62, because when we know a linkgroup in a row all the other ratios of the row can be found by multiplication; and when all the ratios of any row of the table are known any one of the $n(n-1)(n-2)/2$ ratios of the table can be found by repeated applications of Theorem VIII. Next, the question, In which of the n rows of the table shall we try to find a linkgroup? arises. To which the answer is, in one of those rows designated by the link numerals that Groups I and II have in common; that is, therefore, in either row 1 or row 4. Naturally, of these two rows, we choose row 1, because three ratios in row 1 are known already, namely, the basic ratios of Group I. To get a linkgroup in row 1 we must find two more ratios in that row. And let the student observe here that these two additional ratios for row 1 cannot be found from the ratios of Group I by multiplication because this group does not involve either link 2 or link 3. The two additional ratios to make a linkgroup in row 1 must, therefore, be found from the ratios of Group II.

And now, in the first place, we observe that ratio N_{12}/N_{32} of Group II contains link 1, so that we know that by one application of Theorem VIII we can go from this ratio in row 2 to N_{21}/N_{31} in row 1.

How do we know this? As follows: Ratios N_{12}/N_{32} and N_{21}/N_{31} involve the same three links, 1, 2, 3, and these three links, being links of the mechanism or gear train with which we are dealing, are undergoing instantaneous, complanar motion; therefore by Theorem VIII there are three cubic angular velocity ratios associated with these three rigid bodies at each and every instant of their motion. The three ratios for these links are

$$\frac{N_{21}}{N_{31}}, \quad \frac{N_{12}}{N_{32}}, \quad \frac{N_{13}}{N_{23}}.$$

Of these we know one, namely N_{12}/N_{32}; a value for each of the other two can therefore be easily found, as follows, using Fig. X-25a. By Theorem VIII we have

$$\frac{N_{12}}{N_{32}} = -\frac{13\text{-}32}{13\text{-}12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and by equation (4), Group II, we have also

$$\frac{N_{12}}{N_{32}} = -\frac{Z_3}{Z_1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

therefore, equations (6) and (7),

$$\frac{13\text{-}32}{13\text{-}12} = \frac{Z_3}{Z_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From equation (8) we see that linear segment *13-32*, Fig. X'-25a, is proportional to Z_3 and segment *13-12* to Z_1; and hence that segment *12-23* is proportional to $Z_1 + Z_3$. Again, by Theorem VIII

$$\frac{N_{21}}{N_{31}} = \frac{12\text{-}31}{23\text{-}21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

and

$$\frac{N_{13}}{N_{23}} = \frac{12\text{-}23}{12\text{-}13}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

but from Fig. X'-25a

$$\frac{23\text{-}31}{23\text{-}21} = \frac{Z_3}{Z_1 + Z_3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

and

$$\frac{12\text{-}23}{12\text{-}13} = \frac{Z_1 + Z_3}{Z_1}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

therefore

$$\frac{N_{21}}{N_{31}} = \frac{Z_3}{Z_1 + Z_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

and

$$\frac{N_{13}}{N_{23}} = \frac{Z_1 + Z_3}{Z_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

Merely to illustrate more thoroughly the process, we have found N_{13}/N_{23}; we do not need it and shall make no use of it.

The student should carefully observe the rules for symbolizing Fig. X'-25a. First, the three velocity-centers 12, 23, 13 must be on the same straight line in order that the Kennedy-Aronhold law may be satisfied; secondly, center 13 must lie between the other two centers because the sign of the given ratio, N_{12}/N_{32}, equation (4), Group II, is negative; thirdly, segments 13-32 and 13-12 must be dimensioned Z_3 and Z_1, respectively, in order that equation (8) may be obeyed; and, fourthly, segment 12-23 must be dimensioned $(Z_1 + Z_3)$ because this segment is evidently the sum of the other two segments. Let the student note, too, that there are two ways to symbolize Fig. X'-25a with velocity-center 13 in both cases between the other two centers but that it is immaterial which of these ways is chosen, for the result is the same in both cases.

Now having found N_{21}/N_{31}, we know four ratios in row 1, namely,

$$\frac{N_{21}}{N_{31}}, \quad \frac{N_{41}}{N_{51}}, \quad \frac{N_{51}}{N_{61}}, \quad \frac{N_{61}}{N_{71}},$$

the first of these being the one whose value we have just found, and the other three the given basic ratios of Group I. One more ratio in row 1 must be found, found too, as we have already pointed out, from the ratios of Group II, not from those of Group I. From the ratios of Group II, by multiplication we can find another ratio in row 2 that involves link 1, namely,

$$\frac{N_{12}}{N_{42}} = -\frac{Z_3}{Z_1} \times \frac{Z'_4}{Z'_5}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

and from this ratio by another application of Theorem VIII with the aid of Fig. X'-25b, reasoning just as we did in deriving N_{21}/N_{31}, we obtain

$$\frac{N_{21}}{N_{41}} = \frac{Z_3 Z'_4}{Z_3 Z'_4 + Z_1 Z'_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

With this ratio known we have a linkgroup in row 1, and can therefore find any ratio of the table. For the sake of convenience, we collect here in one place the ratios of the linkgroup that we now know in row 1:

Linkgroup I

$$\frac{N_{41}}{N_{51}} = -\frac{Z_5}{Z_4}, \quad \frac{N_{51}}{N_{61}} = -\frac{Z_6}{Z'_5}, \quad \frac{N_{61}}{N_{71}} = -\frac{Z_7}{Z_6},$$

$$\frac{N_{21}}{N_{31}} = \frac{Z_3}{Z_1 + Z_3}, \quad \frac{N_{21}}{N_{41}} = \frac{Z_3 Z'_4}{Z_3 Z'_4 + Z_1 Z'_3}.$$

The first three of these five ratios were given as basic ratios, Group I; the fourth and fifth were derived as explained above from the basic ratios of Group II.

We next write down two rows of the table for 7-link trains, rows 1 and 2. These rows are chosen because in one of them we know a linkgroup and in the other are all the ratios whose values we seek, namely, N_{32}/N_{52}, N_{32}/N_{62}, N_{32}/N_{72}.

$\overline{\overline{N_{21}}}/N_{31}$	$\overline{\overline{N_{21}}}/N_{41}$	$\overline{\overline{N_{21}}}/N_{51}$	$\overline{\overline{N_{21}}}/N_{61}$	$\overline{\overline{N_{21}}}/N_{71}$	$\overline{\overline{N_{31}}}/N_{41}$	$\underline{\underline{N_{31}}}/N_{51}$	$\underline{\underline{N_{31}}}/N_{61}$	$\underline{\underline{N_{31}}}/N_{71}$	$\overline{\overline{N_{41}}}/N_{51}$	$\underline{\underline{N_{41}}}/N_{61}$	$\underline{\underline{N_{41}}}/N_{71}$	$\overline{\overline{N_{51}}}/N_{61}$	$\underline{\underline{N_{51}}}/N_{71}$	$\overline{\overline{N_{61}}}/N_{71}$
$\overline{\overline{N_{12}}}/N_{32}$	$\underline{\underline{N_{12}}}/N_{42}$	$\underline{\underline{N_{12}}}/N_{52}$	$\underline{\underline{N_{12}}}/N_{62}$	$\underline{\underline{N_{12}}}/N_{72}$	$\overline{\overline{N_{32}}}/N_{42}$	?$\underline{\underline{N_{32}}}/N_{52}$	?$\underline{\underline{N_{32}}}/N_{62}$	?$\underline{\underline{N_{32}}}/N_{72}$	$\underline{\underline{N_{42}}}/N_{52}$	$\underline{\underline{N_{42}}}/N_{62}$	$\underline{\underline{N_{42}}}/N_{72}$	$\underline{\underline{N_{52}}}/N_{62}$	$\underline{\underline{N_{52}}}/N_{72}$	$\underline{\underline{N_{62}}}/N_{72}$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The ratios in row 1 whose values we know--see Linkgroup I, p. 62--are overscored with a double line; the ratios in this row whose values we must find are underscored with a double line; the three ratios in row 2 whose values are to be found are indicated with question marks; the ratios in row 2 whose values are known are overscored with a double line, as in row 1. Since none of the three ratios in row 2 whose values we are finding contains link 1, there is no ratio in row 1 that contains the *same* three links as any one of the required ratios; therefore we cannot go directly from row 1 to the desired ratios in row 2 by applications of Theorem VIII. We must, instead, first find by multiplication values for all of the ratios of row 1 that contain link 2 and then by as many applications of Theorem VIII as are necessary go from these ratios in row 1 to ratios in row 2. (The number of ratios in a row each of which contains a given link is, it is easy to see, $(n-2)$. In this case the ratios in row 1 that involve link 2 are, it is evident, the first five ratios in the row; and of these five we know the first two--see the basic ratios of Group II. The third we find by multiplying the second by the tenth; the fourth by multiplying the third by the thirteenth; the fifth by multiplying the fourth by the fifteenth. Performing these operations we have

$$\frac{N_{21}}{N_{51}} = \frac{N_{21}}{N_{41}} \times \frac{N_{41}}{N_{51}} = \frac{Z_3 Z'_4}{Z_3 Z'_4 + Z_1 Z'_3} \times \left(-\frac{Z_5}{Z_4}\right)$$

$$= \frac{-Z_3 Z'_4 Z_5}{Z_3 Z'_4 Z_4 + Z_1 Z'_3 Z_4} \quad \ldots\ldots\ldots\ldots \quad (17)$$

$$\frac{N_{21}}{N_{61}} = \frac{N_{21}}{N_{51}} \times \frac{N_{51}}{N_{61}} = \frac{-Z_3Z_4'Z_5}{Z_3Z_4'Z_4 + Z_1Z_3'Z_4} \times \left(-\frac{Z_6}{Z_5'}\right)$$

$$= \frac{Z_3Z_4'Z_5Z_6}{Z_3Z_4'Z_4Z_5' + Z_1Z_3'Z_4Z_5'} \quad \ldots\ldots\ldots\ldots \quad (18)$$

$$\frac{N_{21}}{N_{71}} = \frac{N_{21}}{N_{61}} \times \frac{N_{61}}{N_{71}} = \frac{Z_3Z_4'Z_5Z_6}{Z_3Z_4'Z_4Z_5' + Z_1Z_3'Z_4Z_5'} \times \left(-\frac{Z_7}{Z_6}\right)$$

$$= \frac{-Z_3Z_4'Z_5Z_7}{Z_3Z_4'Z_4Z_5' + Z_1Z_3'Z_4Z_5'} \quad \ldots\ldots\ldots\ldots \quad (19)$$

From the ratios of equations (17), (18), (19), all of which contain link 2, we go by three applications of Theorem VIII to the ratios of row 2 that involve the same three links respectively.

From Theorem VIII, with the aid of figures X'-25c, X'-25d, X'-25e, respectively, we have

$$\frac{N_{12}}{N_{52}} = \frac{15\text{-}52}{15\text{-}12} = \frac{Z_3Z_4'Z_5}{Z_3Z_4'Z_4 + Z_1Z_3'Z_4 + Z_3Z_4'Z_5}; \quad \ldots\ldots\ldots \quad (20)$$

$$\frac{N_{12}}{N_{62}} = -\frac{16\text{-}62}{16\text{-}12} = \frac{-Z_3Z_4'Z_5Z_6}{Z_3Z_4'Z_4Z_5' + Z_1Z_3'Z_4Z_5' - Z_3Z_4'Z_5Z_6}$$

$$= \frac{Z_3Z_4'Z_5Z_6}{Z_3Z_4'Z_5Z_6 - Z_3Z_4'Z_4Z_5' - Z_1Z_3'Z_4Z_5'}, \quad \ldots\ldots \quad (21)$$

and

$$\frac{N_{12}}{N_{72}} = \frac{17\text{-}72}{17\text{-}12} = \frac{Z_3Z_4'Z_5Z_7}{Z_3Z_4'Z_4Z_5' + Z_1Z_3'Z_4Z_5' + Z_3Z_4'Z_5Z_7} \quad \ldots\ldots\ldots \quad (22)$$

And now that we know a linkgroup in row 2, namely,

Linkgroup II

$$\frac{N_{12}}{N_{32}}, \quad \frac{N_{12}}{N_{42}}, \quad \frac{N_{12}}{N_{52}}, \quad \frac{N_{12}}{N_{62}}, \quad \frac{N_{12}}{N_{72}},$$

we can find any of the other ratios in this row by multiplication. Thus

$$\frac{N_{32}}{N_{52}} = \frac{N_{32}}{N_{12}} \times \frac{N_{12}}{N_{52}} = -\frac{Z_1}{Z_3} \times \frac{Z_3Z_4'Z_5}{Z_3Z_4'Z_4 + Z_1Z_3'Z_4 + Z_3Z_4'Z_5}$$

$$= -\frac{Z_1Z_4'Z_5}{Z_3Z_4'Z_4 + Z_1Z_3'Z_4 + Z_3Z_4'Z_5}, \quad \ldots\ldots\ldots \quad (23)$$

$$\frac{N_{32}}{N_{62}} = \frac{N_{32}}{N_{12}} \times \frac{N_{12}}{N_{62}} = -\frac{Z_1}{Z_3} \times \frac{Z_3Z_4'Z_5Z_6}{Z_3Z_4'Z_5Z_6 - Z_3Z_4'Z_4Z_5' - Z_1Z_3'Z_4Z_5'}$$

$$= \frac{Z_1Z_4'Z_5Z_6}{Z_1Z_3'Z_4Z_5' - Z_3Z_4'Z_5Z_6 + Z_3Z_4'Z_4Z_5'} \quad \ldots\ldots\ldots \quad (24)$$

$$\frac{N_{32}}{N_{72}} = \frac{N_{32}}{N_{12}} \times \frac{N_{12}}{N_{72}} = -\frac{Z_1}{Z_3} \times \frac{Z_3Z'_4Z_5Z_7}{Z_3Z'_4Z_4Z'_5 + Z_1Z'_3Z_4Z'_5 + Z_3Z'_4Z_5Z_7}$$

$$= -\frac{Z_1Z'_4Z_5Z_7}{Z_3Z'_4Z_4Z'_5 + Z_1Z'_3Z_4Z'_5 + Z_3Z'_4Z_5Z_7}. \quad \ldots\ldots\ldots \quad (25)$$

The Values of N_{52}, N_{62}, and N_{72}.--To find numerical values for these three required angular velocities we substitute in equations (23), (24), and (25) the numbers given in Fig. X-25 for the numbers of teeth and for N_{32} the value of 400. Thus:

$$N_{52} = -\frac{Z_3Z'_4Z_4 + Z_1Z'_3Z_4 + Z_3Z'_4Z_5}{Z_1Z'_4Z_5} \times N_{32}. \quad \ldots\ldots\ldots\ldots \quad (26)$$

$$= -\frac{20\times25\times50 + 20\times15\times50 + 20\times25\times40}{20\times25\times40} \times 500.$$

$$= -\frac{3}{1} \times 500 = -1500 \text{ Rpm.}$$

$$N_{62} = -\frac{Z_3Z'_4Z_5Z_6 - Z_3Z'_4Z_4Z'_5 - Z_1Z'_3Z_4Z'_5}{Z_1Z'_4Z_5Z_6} N_{32} \quad \ldots\ldots\ldots\ldots \quad (27)$$

$$= -\frac{20\times25\times40\times25 - 20\times25\times50\times60 - 20\times15\times50\times60}{20\times25\times40\times25} \times 500.$$

$$= \frac{19}{5} \times 500 = 1900 \text{ Rpm.}$$

$$N_{72} = -\frac{Z_3Z'_4Z_4Z'_5 + Z_1Z'_3Z_4Z'_5 + Z_3Z'_4Z_5Z_7}{Z_1Z'_4Z_5Z_7} \times N_{32} \quad \ldots\ldots\ldots\ldots \quad (28)$$

$$= -\frac{20\times25\times50\times60 + 20\times15\times50\times60 + 20\times25\times40\times60}{20\times25\times40\times60} \times 500$$

$$= -\frac{3}{1} \times 500 = -1500 \text{ Rpm.}$$

Problem X-32.--Composition of the Train. The train of gears with which we are to deal in this problem is a bibasal sextic, that is to say, a 6-link train of two baselinks, Fig. X-32. It is composed of *n-2 (=4)* cubic spur gear trains: 156, 345, 234, 134; or, looking at it in another way, of one cubic spur-gear train 156 and a unibasal quintic 12345, link 1 being the baselink of the cubic and link 3 the baselink of the quintic. The component cubic and quintic have links 1 and 5 in common. Link 4 of the mechanism, it should be noticed, is a triplex gear. Let it be noted, moreover, that the four links 1, 2, 3, 5 all turn about the same axis relative to each other; in other words, the six velocity axes for these four links coincide with each other as shown in the figure, and thus each of these links has complanar motion relative to each of the others.

How the Mechanism Works.--If the observer stands on the left-hand side of the machine and turns link 2 counterclockwise relative to baselink 1, triplex gear 4 will be compelled to move; but since gear Z''_4, one of the gears of triplex gear 4, meshes gear 1 and since gear 1 is rigidly fastened to baselink 1, gear 4 can move only by rolling around clockwise on the outside of gear 1; in other words, when link 2 is turned about axis 12 relative to baselink 1, link 4 is compelled to turn relative to link 1 about axis 14, that is, about

line of tangency of the two pitch cylinders of gears Z_1 and Z_4''. Now as link 4 rolls around on the outside of gear 1 it pulls or drags baselink 3 around in a clockwise sense with it and also drives gear 5. Gear 5 in turn drives gear 6 which it meshes.

The problem, obviously, is to determine expressions in terms of the *Zs* for the angular velocity ratios

$$\frac{N_{21}}{N_{31}} \text{ and } \frac{N_{21}}{N_{61}},$$

for when these ratios and N_{21} (the given angular velocity) are known it is easy to compute N_{31} and N_{61}, the two required angular velocities.

Derivation of Angular Velocity Ratios N_{21}/N_{31} and N_{21}/N_{61}.--We first write down the *n-2 (=4)* basic ratios that belong to the *n-2 (=4)* component spur-gear trains.

Basic Ratios - Group I

Baselink 1

$$\frac{N_{51}}{N_{61}} = -\frac{Z_6}{Z_5} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Basic Ratios- Group II

Baselink 3

$$\frac{N_{43}}{N_{53}} = -\frac{Z_5'}{Z_4} \quad \ldots (1), \qquad \frac{N_{43}}{N_{23}} = -\frac{Z_2}{Z_4'} \quad \ldots (2), \qquad \frac{N_{13}}{N_{43}} = -\frac{Z_4''}{Z_1} \quad \ldots (3)$$

These two groups of ratios have links 1 and 5 in common; thus we can find a linkgroup of ratios in each of these two rows of the table of angular velocity ratios for sextics. We choose row 1, below, because the desired ratios are in this row. From the *n-2 (=3)* ratios in row 3 we can find three others in this row by multiplication; that is to say, we can find all the other ratios of this row each of which involves some combination of two of the links 1, 2, 4, 5. And *n-3 (=3)* of these combinations will involve link 1. The third basic ratio of Group II is one of these *n-3 (=3)* ratios that involve link 1. The other two are found by multiplying ratios (1) and (2) of Group II by (3) of this group. Thus the ratios in row 3, each of which involves link 1, are:

Minor Linkgroup - Row 3

$$\frac{N_{13}}{N_{43}} = -\frac{Z_4''}{Z_1}, \qquad \frac{N_{13}}{N_{23}} = \frac{Z_2 Z_4''}{Z_1 Z_4'}, \qquad \frac{N_{13}}{N_{53}} = \frac{Z_4'' Z_5'}{Z_1 Z_4}$$

Now by three applications of Theorem VIII with the aid of suitable figures we find from these three ratios the minor linkgroup

$$\frac{N_{21}}{N_{31}}, \quad \frac{N_{31}}{N_{41}}, \quad \frac{N_{31}}{N_{51}}$$

in row 1. The values of these three ratios are thus:

Minor Linkgroup - Row 1

$$\frac{N_{31}}{N_{41}} = \frac{Z_4''}{Z_1 + Z_4''} \quad \ldots (1), \qquad \frac{N_{21}}{N_{31}} = \frac{Z_2 Z_4'' - Z_1 Z_4'}{Z_2 Z_4''} \quad \ldots (2), \qquad \frac{N_{31}}{N_{51}} = \frac{Z_4'' Z_5'}{Z_4'' Z_5' - Z_1 Z_4} \quad \ldots (3)$$

The basic ratio $N_{51}/N_{61} = -Z_6/Z_5$ of Group I added to these gives us a linkgroup in row 1 with which all the other ratios of row 1, among which the two required ratios are inclined, can be found by multiplication. Row 1 of the table for sextics is:

$\overline{\frac{N_{21}}{N_{31}}}$,	$\frac{N_{21}}{N_{41}}$,	$\frac{N_{21}}{N_{51}}$,	$\frac{N_{21}}{N_{61}}$,	$\overline{\frac{N_{31}}{N_{41}}}$,	$\overline{\frac{N_{31}}{N_{51}}}$,	$\frac{N_{31}}{N_{61}}$,	$\frac{N_{41}}{N_{51}}$,	$\frac{N_{41}}{N_{61}}$,	$\overline{\frac{N_{51}}{N_{51}}}$
1	2	3	4	5	6	7	8	9	10

in which the ratios that we know are overscored and those to be determined marked with a question mark.

Multiplying the 6th by the 10th gives the 7th

$$\frac{N_{31}}{N_{61}} = \frac{N_{31}}{N_{51}} \times \frac{N_{51}}{N_{61}}$$

$$= -\frac{Z''_4 Z'_5}{Z''_4 Z'_5 - Z_1 Z_4} \times \frac{Z_6}{Z_5}$$

$$= \frac{Z''_4 Z'_5 Z_6}{Z_1 Z_4 Z_5 - Z''_4 Z_5 Z'_5}; \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

and then multiplying the first by the 7th gives the 4th, one of the required ratios. Thus, performing these operations

$$\frac{N_{21}}{N_{61}} = \frac{N_{21}}{N_{31}} \times \frac{N_{31}}{N_{61}}$$

$$= \frac{Z_2 Z''_4 - Z_1 Z'_4}{Z_2 Z''_4} \times \frac{Z''_4 Z'_5 Z_6}{Z_1 Z_4 Z_5 - Z''_4 Z_5 Z'_5}$$

$$= \frac{Z_2 Z''_4 Z'_5 Z_6 - Z_1 Z'_4 Z'_5 Z_6}{Z_1 Z_2 Z_4 Z_5 - Z_2 Z''_4 Z_5 Z'_5}. \quad \ldots\ldots\ldots\ldots \quad (5)$$

The value of N_{21}/N_{31}, the other one of the two required ratios of the problem appears in the minor linkgroup of row 1, equation (2) of that group.

The Values of N_{31} and N_{61} for the Given Conditions of the Problem.--To find N_{31} we have

$$N_{31} = \frac{Z_2 Z''_4}{Z_2 Z''_4 - Z_1 Z'_4} \times N_{21}$$

$$= \frac{\overset{6}{\cancel{54}} \times \overset{2}{\cancel{20}}}{\underset{1}{54} \times \underset{2}{20} - \underset{9}{81} \times \underset{3}{30}} \times (-20)$$

$$= \frac{4}{5} \times 20 = 16 \text{ Rpm.}$$

To find N_{51} we have, equation (5)

$$N_{61} = \frac{Z_1 Z_2 Z_4 Z_5 - Z_2 Z''_4 Z_5 Z'_5}{Z_2 Z''_4 Z'_5 Z_6 - Z_1 Z'_4 Z'_5 Z_6} \times N_{21}$$

$$N_{61} = \frac{\overset{3}{\underset{9}{81}} \times \overset{4}{\underset{8}{54}} \times \overset{2}{\underset{6}{88}} \times \overset{6}{\underset{2}{36}} - \overset{2}{\underset{6}{54}} \times \overset{2}{\underset{3}{20}} \times \overset{6}{\underset{4}{36}} \times 54}{\underset{6}{54} \; \underset{2}{20} \; \underset{8}{54} \; 48 - \underset{9}{81} \; \underset{3}{30} \; 54 \; \underset{8}{48}} \times (-20)$$

$$= -\frac{60}{20}(-20) = 60 \text{ Rpm.}$$

Problem X-34. Description of the Mechanism. The mechanism, Fig. X-34, with which this problem deals is part of a water-wheel governor. It is essentially a 7-link mechanism despite the fact that eight links appear in the figure, because one of the links, 9 or 10, is kinematically superfluous. Shaft 2 of cone pulley 2 is belted to the water wheel, not shown in the figure, and shaft 6 operates (or at least so we may suppose) the flume gate (not shown) through a spur-gear pinion on shaft 6 and a vertical rack fastened to the gate. The belt shipper, not shown in the figure, is connected by suitable mechanism to the sleeve of a centrifugal governor which is belted to the water wheel. Normally, when the work that is required of the water wheel is exactly equal to the energy delivered to it by the water, shaft 6 and the flume gate that it operates remain at rest; but when the energy-output required of the wheel is greater than the amount of energy the water is delivering to it, the speed of the wheel will fall and thus also that of cone pulley 2 and the centrifugal governor, since, being belted to the wheel, the speed of each of these is always directly proportional to the speed of the wheel. But when the speed of the centrifugal governor falls, the sleeve of the governor moves down on its spindle, and this motion of the sleeve causes the belt shipper to move the belt away from its neutral position on the pulleys to a position that will give shaft 6 motion in the right sense to open the gate somewhat, and thus supply the wheel with the additional water and therefore with the additional energy needed. And when equilibrium between the energy-output required of the wheel and the energy-input supplied by the water is established, shaft 6 and thus also the gate will once more come to rest. If, on the other hand, the work demanded of the wheel is decreased, the belt will be moved in the other direction from its neutral position, and the gate will be moved downward, thus supplying less water and therefore less energy to the wheel.

It will appear from the analysis that is to follow that, with the mechanism geared up as shown in the figure a *crossed* belt would not be suitable if it is to be used as a water-wheel governor, for geared-up as the mechanism is and with a crossed belt, there would be no position of the belt on the pulleys that would cause shaft 6 to be at rest relative to link 1. (The student should be sure to satisfy himself of the truth of this statement.) To make a crossed belt suitable any one of the following three changes would have to be made in the gearing: (1) extend shaft 2 upward and mesh bevel-gear 2 on the upper side of bevel-gear 5; (2) extend the sleeve to which bevel-gear 5 is fastened to the right and mesh bevel-gear 5 on the right-hand side of gear 2; (3) extend the sleeve to which bevel-gear 4 is fastened to the left and mesh 4 with 3 on the left side of the latter.

It should be noticed that four links of this mechanism--1, 4, 5, 6, 9, 10--form the well-known bevel-gear differential used on the rear axle of automobiles. The gears of this differential are Z'_4, Z'_5, Z_9, Z_{10}, the former two of which are exactly alike and the latter two of which are also exactly alike. In automobile practice the first two gears are always much larger than the other two. It should be noticed, too, that one of the latter is kinematically superfluous, that is to say, the four links 1, 4, 5, 6 would be subjected to exactly the same kinematic constraint with one of these gears omitted as with it in place. The purpose in having two or more "planet" pinions like 9 and 10 in the differential is twofold: (1) for effecting dynamic balance (an important consideration at high speeds); (2) for decreasing the load on the teeth, thus allowing all of the gears to be made smaller and the mechanism more compact and less expensive.

Solution of Problem X-34.--The first thing that strikes one's attention when one starts to analyze this very interesting mechanism is that links 1, 4, 5, 6, the links that help to form the differential, all turn about the same permanent axis of rotation relative to each other, and thus that each of these parts has complanar motion relative to each of the others. The next thing that strikes the attention is that links 4 and 5 must turn at the same speed in opposite senses relative to link 6, and that therefore

$$\frac{N_{46}}{N_{56}} = \frac{N_{64}}{N_{65}} = -1. \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

(The student can easily satisfy himself that this relation is correct either by examining Fig. X-34 or the model in the lecture room.)

Now because links 1, 4, 6 are undergoing complanar, relative motion, Theorem IX holds, and thus we have the well-known equation

$$N_{61} = N_{41} + N_{64}. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

And because links 1, 5, 6 are also undergoing complanar motion we have, similarly,

$$N_{61} = N_{51} + N_{65}. \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

Adding equations (2) and (3), remembering that equation (1) shows that N_{64} is equal and opposite to N_{65}, we get

$$2N_{61} = N_{41} + N_{51}$$

or

$$N_{61} = \frac{N_{41} + N_{51}}{2}. \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

Equations (1) and (4) express symbolically the fundamental kinematic laws of the bevel-gear differential, and all kinematic problems that arise in connection with this mechanism can be solved by their use.

Examining equation (4), which involves not three--as equations (2) and (3) do--but four links, we see that it asserts that the speed of shaft 6 relative to link 1 is always half the *algebraic* sum of the speeds of links 4 and 5 relative to link 1. This means, for one thing, that if links 4 and 5 are turned in the same sense and at the same speed relative to link 1, shaft 6 will turn at the same speed relative to link 1 as 4 and 5 do; in other words, links 4, 5, and 6 will for these conditions turn as one rigid body. For another thing, it means that if links 4 and 5 turn, however fast or slowly, at the same speed but in opposite senses relative to link 1, shaft 6 will remain at rest relative to link 1.

Now most kinematic problems concerning the water-wheel governor mechanism in which this differential is used require a knowledge of the value of the ratio N_{61}/N_{21} or a knowledge of the ratio N_{61}/N_{31}. For the first of these ratios we can easily obtain an expression in terms of the numbers of teeth on the four gears and the diameter of the two cone pulleys (at the place where the belt runs) by dividing both sides of equation (4) by N_{21}; thus

$$\frac{N_{61}}{N_{21}} = \frac{1}{2}\left(\frac{N_{51}}{N_{21}} + \frac{N_{41}}{N_{21}}\right). \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

From bevel-gears 2 and 5, moreover, paying no attention to the sign of the right-hand side of the equation, which will be attended to in detail presently, we know

$$\frac{N_{51}}{N_{21}} = \frac{Z_2}{Z_5}; \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (6)$$

and from cone pulleys 2 and 3 and bevel-gears 3 and 4, again, for the moment, disregarding signs,

$$\frac{N_{31}}{N_{21}} = \frac{D_2}{D_3}, \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (7)$$

and

$$\frac{N_{41}}{N_{31}} = \frac{Z_3}{Z_4}. \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (8)$$

From equations (7) and (8)--

$$\frac{N_{41}}{N_{21}} = \frac{N_{41}}{N_{31}} \times \frac{N_{31}}{N_{21}} = \frac{Z_3 D_2}{Z_4 D_3}. \quad \ldots \ldots \ldots \ldots \ldots \quad (9)$$

And then substituting from equations (6) and (9) in equation (5) there results the relation that expresses the ratio N_{61}/N_{21} in terms of the numbers of teeth on the four gears and the diameters of the two cone pulleys

$$\frac{N_{61}}{N_{21}} = \frac{1}{2}\left(\frac{Z_2}{Z_5} + \frac{D_2 Z_3}{D_3 Z_4}\right). \quad \ldots \ldots \ldots \ldots \ldots \quad (10)$$

But to make equation (5) or its equivalent equation (10) fully useful in analyzing the mechanism with which we are dealing it is necessary to have a rule for fixing the sign of each of the three terms that enter into it. And to devise such a rule it is necessary to have two conventions: (1) a convention for specifying the position of the observer relative to the machine when distinguishing clockwise from counterclockwise rotation of cone pulleys 2 and 3; (2) a second convention for specifying the position of the observer when distinguishing clockwise rotation from counterclockwise rotation for the relative motion of any two of the four links 1, 4, 5, 6.

Convention I.--To distinguish clockwise from counterclockwise motion of pieces 2 and 3 relative to link 1 the observer stands in front of the machine and looks towards it. (The view of the machine shown is supposed to be a top view.)

Convention II.--To distinguish clockwise from counterclockwise motion of any one of pieces 1, 4, 5, 6 relative to any other one of these pieces the observer stands on the left of the machine and looks along velocity-axis 16, from left to right.

For both cases, moreover, clockwise rotation will be regarded as positive and counterclockwise rotation negative.

Now an examination of equation (5) shows: (1) That the sign of N_{21} can have no effect on the sign of any of the terms of the equation, for N_{21} occurs in each of these terms, and that, therefore, we may consider N_{21} to be positive whether cone pulley 2 turns clockwise or counterclockwise relative to baselink 1; (2) that the numerical value and sign of N_{61} depend on the numerical values and signs of N_{41}/N_{21} and N_{51}/N_{21}, because, equation (4), the magnitude and sense of the motion of link 6 relative to link 1 depend on the magnitudes and senses of the motions of links 4 and 5 relative to baselink 1.

An examination of the mechanism, keeping in mind Conventions I and II, shows the truth of the following four statements:

1. That if the belt is *crossed* and cone pulley 2 is turned *counterclockwise*, links 4 and 5 will both turn clockwise; that consequently N_{41} and N_{51} will both be *positive*, and equation (5) will become for this case

$$\frac{N_{61}}{N_{21}} = \frac{1}{2}\left(\frac{N_{51}}{N_{21}} + \frac{N_{41}}{N_{21}}\right) . \quad . . . . . \quad (11)$$

2. That if the belt is *crossed* and cone pulley 2 turns *clockwise* relative to link 1, links 4 and 5 will both turn counterclockwise relative to link 1; that therefore N_{41} and N_{51} will both be *negative*, and that thus equation (5) becomes for this case

$$\frac{N_{61}}{N_{21}} = \frac{1}{2}\left(-\frac{N_{51}}{N_{21}} - \frac{N_{41}}{N_{21}}\right)$$

$$= -\frac{1}{2}\left(\frac{N_{51}}{N_{21}} + \frac{N_{41}}{N_{21}}\right) . \quad . . . . . \quad (12)$$

3. That if the belt is *open* and cone pulley 2 turns *counterclockwise*, link 5 will then turn clockwise and link 4 counterclockwise; that therefore N_{41} will be *negative* and N_{51} and *positive*, and that for this case accordingly equation (5) becomes

$$\frac{N_{61}}{N_{21}} = \frac{1}{2}\left(\frac{N_{51}}{N_{21}} - \frac{N_{41}}{N_{21}}\right) . \quad . . . . . \quad (13)$$

4. That if the belt is *open* and cone pulley 2 turns *clockwise*, link 5 will turn counterclockwise and link 4 clockwise; that therefore N_{41} will be positive and N_{51} negative, and that for this case equation (5) will thus take the form

$$\frac{N_{61}}{N_{21}} = \frac{1}{2}\left(-\frac{N_{51}}{N_{21}} + \frac{N_{41}}{N_{21}}\right)$$

$$= -\frac{1}{2}\left(\frac{N_{51}}{N_{21}} - \frac{N_{41}}{N_{21}}\right) . \quad . . . . . \quad (14)$$

Now quite obviously, making use of the artifice of the double sign, equations (11), (12), (13), and (14) can be combined into a single one, that is, into

$$\frac{N_{61}}{N_{21}} = \pm\frac{1}{2}\left(\frac{N_{51}}{N_{21}} \pm \frac{N_{41}}{N_{21}}\right) , \quad . . . . . \quad (15)$$

for which, evidently, we have the following rule of signs:

Rule of Signs for Equation (15).--Take the *positive* or the *negative* sign inside the parenthesis according as the belt is *crossed* or *open*; take the *positive* or the *negative* sign outside the parenthesis according as cone pulley 2 turns *counterclockwise* or *clockwise*.

Moreover, employing the double sign, we may also generalize equation (10) as follows:

$$\frac{N_{61}}{N_{21}} = \pm\frac{1}{2}\left(\frac{Z_2}{Z_5} \pm \frac{D_2 Z_3}{D_3 Z_4}\right) . \quad . . . . . \quad (16)$$

And with this equation thus generalized we can solve with the greatest ease any kinematic problem that can arise concerning the water-wheel governor mechanism of Fig. X-34. But the student must remember that the rule of signs for using equation (16) is valid only if the mechanism is geared up as shown in the figure. If, to put the matter differently, the mechanism is geared up in any one of the other three ways discussed in the description of the machine, the rule of signs for using equation (16) would have to be modified accordingly. This is an important matter and the student should be sure to examine the statement just made and determine what modification the rule of signs would have to undergo for each of these three different ways of gearing the mechanism.

We shall next show how to use equation (16) to solve some problems concerning this water-wheel-governor mechanism.

Problem X-34₁.--Where must the belt run on the two pulleys and must the belt be *open* or *crossed* in order that when cone pulley 2 is turned *counterclockwise* at the rate of 300 rpm. shaft 6 will be driven *clockwise* at the rate of 1100 rpm?

Solution.--For N_{61} and N_{21} in equation (16) we must, it is evident, substitute +1100 and +300, respectively. (The 1100 is positive, the student should remember because the problem specifies that shaft 6 is to turn clockwise relative to link 1 (see Convention II); the 300 is positive, he should remember, because equation (16) is derived on the assumption that the sense of rotation of cone pulley 2 relative to baselink 1, that is, the sign of N_{21}, can have no effect on the signs of its individual terms.) The positive sign in front of the parenthesis must be taken because cone pulley 2 turns counterclockwise (see the Rule of Signs for Equation 15). The sign inside the parenthesis cannot be determined until we have made the numerical substitutions and are thus in a position to know the relative magnitudes of the two *known* terms--the terms inside the parenthesis--of the equation. Making these substitutions we get

$$\frac{1100}{300} = \frac{11}{3} = \frac{1}{2}\left(\frac{80}{30} \pm \frac{D_2}{D_3} \times \frac{56}{16}\right)$$

$$= \frac{1}{2}\left(\frac{8}{3} \pm \frac{D_2}{D_3} \times \frac{7}{2}\right),$$

that is,

$$\frac{22}{3} = \frac{8}{3} \pm \frac{D_2}{D_3} \times \frac{7}{2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

Considering this equation, we see at once that the minus sign and therefore the open belt to which it corresponds are inadmissable, because, clearly there is no positive--and it must be positive--value of D_2/D_3 which when multiplied by 7/2 and the product subtracted from 8/3 will leave a remainder of 22/3. We must therefore take the plus sign and with it the crossed belt. Adopting this sign and solving the equation for D_2/D_3, we get

$$\frac{D_2}{D_3} = \frac{4}{3}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

that is to say, the belt must run on the cone pulleys at the place where the ratio of the two diameters is as 4 to 3.

Of course the result that we have just obtained doesn't tell us, the student must understand, what the sizes of the two diameters are, nor does it give us the position of the belt on the pulleys. To find D_2 and D_3 and the position of the belt on the pulleys we must have another relation between D_2 and D_3. Since the belt is crossed, the two pulleys may be cones of revolution, (see Art. 13-12, p. 331 of the text book) and for cones of revolution the other relation between D_2 and D_3 is thus

$$D_2 + D_3 = a\ constant. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

And if the constant of equation (19) is known, as it would be, of course, if the dimensions of the pulleys were known, equations (18) and (19) would give D_2 and D_3; and from either of them the position of the belt could be found.

Problem X-34₂.--If (1) the belt is open, (2) $D_2/D_3 = 1$, and (3) cone pulley 2 turns *clockwise* at the rate of 600 rpm., at what rate and in which sense relative to link 1 will shaft 6 turn?

Solution.--Making use of formula (16) and the rule of signs we must, it is evident, take the negative sign inside the parenthesis because the belt is *open*; moreover, the negative sign outside the parenthesis must be chosen because cone pulley 2 turns *clockwise* relative to link 1. Thus, substituting the numerical values in equation (16), we have

$$\frac{N_{61}}{600} = -\frac{1}{2}\left(\frac{80}{30} - 1 \times \frac{56}{16}\right)$$

$$= -\frac{1}{2}\left(\frac{8}{3} - \frac{7}{2}\right)$$

$$= -\frac{1}{2}\left(\frac{16}{6} - \frac{21}{6}\right) = -\frac{1}{2}\left(-\frac{5}{6}\right) = \frac{5}{12},$$

and consequently

$$N_{61} = 250 \text{ Rpm., } clockwise.$$

Problem X-34$_3$.--(1) Where must the belt run on the pulleys and must it be open or crossed in order for shaft 6 to remain at rest? (2) Determine, moreover, whether for this position of the belt (assuming that such a position exists) shaft 6 would remain at rest for all speeds of cone pulley 2, or only for some particular speed or speeds.

Solution.--(1) First, we observe here that the left-hand side of equation (16) must be zero, because by hypothesis N_{61} is zero, and that thus also the right-hand side of the equation must likewise be zero. Next, we observe that the plus sign inside the parenthesis, and therefore the crossed belt, is inadmissable here because the sum of two *positive* quantities--the quantities inside the parenthesis--cannot be zero. We observe in the third place that the sign in front of the parenthesis is immaterial, because it is obvious that it has no effect on either the sign or the magnitude of D_2/D_3, the quantity whose value is sought.

Taking, therefore, the minus sign (open belt) inside the parenthesis; either sign in front of the parenthesis, and $N_{61} = 0$, equation (16) gives

$$0 = \frac{1}{2}\left(\frac{8}{3} - \frac{D_2}{D_3} \times \frac{7}{2}\right)$$

that is,

$$\frac{D_2}{D_3} = \frac{16}{21};$$

in other words, with the mechanism geared up as shown in the figure and the cone pulleys connected with an open belt, the belt must run at a place on the pulleys where the ratio of the diameters is 16/21, if shaft 6 is to stand still.

(2) The right-hand side of equation (16) will, the elementary principles of algebra show, be zero as long as N_{61} is zero, regardless of the finite value of N_{21}; thus the value of the ratio D_2/D_{31} and thus also the position of the belt, is unaffected by the value of N_{21}; in other words, at whatever speed and in whichever sense cone pulley 2 turns relative to link 1, the belt must run at a place on the pulleys where the ratio $D_2/D_3 = 16/21$ in order that shaft 6 may remain at rest.

Problem X-34$_4$.--If: (1) a clockwise torque (twisting or turning moment) of 1100 ft. lbs. acts on cone pulley 2, (2) the efficiency of the machine is 75%, (3) the belt is crossed, and (4) the belt is so located on the pulleys that $D_2/D_3 = 4/3$; show that a counterclockwise torque of 225 ft. lbs. will be developed on shaft 6.

Solution.--If friction at the various bearings is neglected, that is to say, if we suppose the efficiency of the machine to be 100%, the energy put into the machine in a minute at cone pulley 2, the driver, must be exactly equal in amount to that developed at the driven shaft 6. But the energy put into the machine in a minute at cone pulley 2 is

$$E_{12} = 2\pi N_{21} R_2 F_{12}, \quad \ldots\ldots\ldots\ldots\ldots \quad (20)$$

where R_2 in feet is the lever-arm at which force F_{12} in lbs. acts on cone pulley 2. And the energy developed in a minute at shaft 6 is, similarly,

$$E_{16} = 2\pi N_{61} R_6 F_{16}. \quad \ldots\ldots\ldots\ldots\ldots \quad (21)$$

Equating E_{12} to E_{16}, we get

$$\frac{N_{21}}{N_{61}} = \frac{R_6 F_{16}}{R_2 F_{12}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (22)$$

But

$$R_6 F_{16} = T_{16}, \quad \ldots\ldots\ldots\ldots\ldots \quad (23)$$

and

$$R_2 F_{12} = T_{12}, \quad \ldots\ldots\ldots\ldots\ldots \quad (24)$$

where T_{16} is the torque or twisting moment in ft. lbs. that force F_{16} exerts on shaft 6, and T_{12}, likewise, the torque or twisting moment in ft. lbs. that force F_{12} exerts on cone pulley 2. Thus

$$\frac{N_{21}}{N_{61}} = \frac{T_{16}}{T_{12}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (25)$$

Now combining equations (25) and (16), we have finally

$$\frac{T_{12}}{T_{16}} = \pm\frac{1}{2}\left(\frac{Z_2}{Z_5} \pm \frac{D_2}{D_3}\frac{Z_3}{Z_4}\right). \quad \ldots\ldots\ldots\ldots\ldots \quad (26)$$

Equation (26) takes no account of friction losses, however, and thus must be modified by the introduction of a coefficient of efficiency. The amount of energy put into a machine at the driver must always be greater than the amount taken out in the form of useful work, because part of the energy put in is wasted at the bearings in friction. If we let E'_{61} represent the energy developed in a minute at shaft 6 when friction losses at all the bearings are taken into account, and E_{12} the energy put into the machine at the driver 2 in the same time, then

$$\frac{E'_{16}}{E_{12}} = \eta \quad \ldots\ldots\ldots\ldots\ldots \quad (27)$$

is the efficiency of the machine. But since, equation (20),

$$E_{12} = 2\pi N_{21} R_2 F_{12},$$

and

$$E'_{16} = 2\pi N_{61} R_6 F'_{16}, \quad \ldots\ldots\ldots\ldots\ldots \quad (29)$$

in the latter of which F'_{16} is the force--less in amount, of course, than F_{16} of equation (21)--developed on shaft 6 when friction at all of the bearings is taken into account, we have (from equations (27), (28), and (29))

$$\frac{2\pi N_{61} R_6 F'_{16}}{2\pi N_{21} R_2 F_{12}} = \eta; \quad \ldots\ldots\ldots\ldots\ldots \quad (30)$$

from which, substituting T'_{16} for $R_6 F'_{16}$ and T_{12} for $R_2 F_{12}$,

$$\frac{N_{21}}{N_{61}} = \frac{R_6 F'_{16}}{\eta R_2 F_{12}} = \frac{T'_{16}}{\eta T_{12}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (31)$$

Finally, combining equations (31) and (16), we get the expression that connects the torque acting on cone pulley 2 to the torque acting on shaft 6, taking into account friction at all rubbing surfaces:

$$\frac{T_{12}}{T'_{16}} = \pm \frac{1}{2\eta} \left(\frac{Z_2}{Z_6} \pm \frac{D_2 Z_3}{D_3 Z_4} \right). \quad \ldots\ldots\ldots\ldots\ldots \quad (32)$$

Comparing equations (26) and (32) we see that (32) is derived from (26) by (1) substituting T'_{16} for T_{16}, and (2) introducing the coefficient of efficiency, η, into the latter as a multiplier of T_{12} or as a divisor of T'_{16}. (5he student will do well to remember these facts.

Problem X-34_4 is now easily solved by substituting the given data in equation (32).

Since the driving torque (twisting or turning moment), T_{12}, acting on cone pulley 2, is clockwise, cone pulley 2 turns clockwise and the sign outside the parenthesis of equation (32) is negative. And since the belt is crossed the sign inside the parenthesis is positive; so that (32) reduces to

$$\frac{1100}{T'_{16}} = -\frac{1}{.75 \times 2} \left(\frac{8}{3} + \frac{4}{3} \times \frac{7}{2} \right) = -\frac{44}{9};$$

$$T'_{16} = -225 \text{ ft. lbs., counterclockwise.}$$

Problem X-39.--Description of the Mechanism. The mechanism--a most interesting one--with which we are now concerned I call the MacCord Unibasal Spheroquintic. The generalized form of it was invented many years ago by the late Professor C. W. MacCord of Stevens Institute. But the mechanism is sometimes called Humpage's gear train, notwithstanding the fact that Professor MacCord had described the general form of it long before Humpage made use of it in lathe design. The prefix *sphero* of the word spheroquintic indicates that the mechanism is spheric, not complanar; nevertheless four of the five links of the train have complanar motion relative to each other. These links are 1, 2, 3, 4. The fifth link has spheric motion relative to each of the others. The mechanism is *unibasal* because it has only one baselink, link 3. At first glance it will seem to the student that this statement is not correct; it will seem, that is, that the mechanism is at least bibasal; but a closer examination of the device will bring out the fact that link 3 is the only link of the five that is paired to at least two other links by lower pairing. It is only because the mechanism is so designed that velocity axes 13, 23, and 34 coincide that it appears to have more than one baselink. If the student will sketch it--and by all means he should do so--with this special relation avoided, he will easily see that link 3 is the only baselink in the train. See Fix. X-39_1.

Without an explanation of the matter, the student will perhaps wonder why we speak of this train as having five links when six are clearly shown in the figure. The reason is that one of the links, 5 or 6, is *kinematically* superfluous; that is to say, the relative motions of links 1,2,3, and 4 would be just as perfectly constrained with only one of

these two links present in the mechanism as with both of them present. Two or more such links as 5 and 6 are used, instead of just one, to give dynamic balance and to reduce the load on the teeth of the gears--excellent reasons, both of them.

Note.--If we were dealing with the more generalized form of this mechanism in which velocity axes 13, 23, and 34 do not coincide the method that we shall use to find the expression for the ratio N_{21}/N_{41} would not be applicable, because then links 1, 2, 3, and 4, the links involved in the required ratio, would not have complanar motion relative to each other. Since these axes do coincide, however, links 1, 2, 3, and 4 all have complanar relative motion and all the laws that we have learned for the complanar motion of rigid bodies hold here for the relative motions of these four links. In other words, we can deal with the twelve cubic angular velocity ratios of these four links just as if the links formed a unibasal quartic.

Statement of Problem X-39.--The problem as we shall solve it consists of two parts: (a) If drive shaft 2 turns clockwise (when viewed from the right-hand side of the device) 2500 rpm. relative to barrel 1, at what rate and in which sense relative to link 1 will shaft 4 turn? (b) If the efficiency of the mechanism is η (=90%) and a clockwise torque of T_{12} (=100) ft. lbs. is applied to shaft 2, what torque (=T_{14}) acting on shaft 4 will torque T_{12} produce?

To solve the first part of the problem we must express the angular velocity ratio N_{21}/N_{41} in terms of the numbers of teeth on the five gears; to solve the second part we must express T_{14} in terms of η, T_{12}, and N_{21}/N_{41}.

(a) *Derivation of Angular Velocity Ratio* N_{21}/N_{41}.--The basic ratios for the quintic, paying at present no attention to signs, are:

$$\frac{N_{23}}{N_{53}} = \frac{Z_5}{Z_2} \quad . \quad . \quad . \quad (1), \qquad \frac{N_{53}}{N_{43}} = \frac{Z_4}{Z_5'} \quad . \quad . \quad . \quad (2), \qquad \frac{N_{13}}{N_{53}} = \frac{Z_5}{Z_1}, \quad . \quad . \quad (3),$$

which, as we see from the symbolism, form a linkgroup in row 3. But the required ratio, N_{21}/N_{41}, lies in row 1; hence we must find a linkgroup in row 1; but to find a linkgroup in row 1 we must first find $n-2$ $(=3)$ ratios in row 3 each of which involves link 1. One of these we already have--basic ratio N_{13}/N_{53}. The other two we find by multiplying (2) by (3) by the reciprocal of (1); accordingly

$$\frac{N_{13}}{N_{43}} = \frac{N_{13}}{N_{53}} \times \frac{N_{53}}{N_{43}} = \frac{Z_5}{Z_1}\frac{Z_4}{Z_5'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{N_{13}}{N_{23}} = \frac{N_{13}}{N_{53}} \times \frac{N_{53}}{N_{23}} = \frac{Z_5}{Z_1} \times \frac{Z_2}{Z_5} = \frac{Z_2}{Z_1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

thus the linkgroup of ratios in row 3 each of which involves link 1 is

$$\frac{N_{13}}{N_{23}}, \quad \frac{N_{13}}{N_{43}}, \quad \frac{N_{13}}{N_{53}}.$$

It is now necessary to observe that we cannot go from ratio N_{13}/N_{53} in row 3, a ratio of our new linkgroup in this row, to ratio N_{31}/N_{51} in row 1 by Theorem VIII, because this theorem is not applicable to the relative motions of the three links involved in any ratio that contains link 5, for, as has already been stated, link 5 has spheric and not complanar motion relative to all the other links of the mechanism except link 3. Fortunately, however, it is not necessary to have the value of N_{31}/N_{51} to find the value of the desired ratio N_{21}/N_{41}, for this ratio, as we see does not involve link 5.

In order to go from N_{13}/N_{43} and N_{13}/N_{23} in row 3 to N_{31}/N_{41} and N_{21}/N_{31}, respectively, in row 1 the signs of the two former must of course first be fixed. This we do not do by convention but by inspecting the figure. Let the observer imagine himself standing on the

right of the machine and looking from right to left at it. Hold baselink 3 stationary and turn shaft 2 clockwise. Then link 1 will, it is obvious, be forced to turn counterclockwise, and thus the sign of N_{13}/N_{23}, equation (5), is negative. Again, if we hold baselink 3 stationary and turn link 1 clockwise, link 4 will turn clockwise; so that the sign of N_{13}/N_{43}, equation (4), is positive.

Applying in the usual way Theorem VIII to equations (4) and (5), and remembering that the sign of the right-hand side of the former is positive and of the latter negative, we get

$$\frac{N_{31}}{N_{41}} = \frac{Z_4 Z_5}{Z_4 Z_5 - Z_1 Z'_5} \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

$$\frac{N_{21}}{N_{31}} = \frac{Z_1 + Z_2}{Z_2}; \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

and then from (6) and (7) by multiplication

$$\frac{N_{21}}{N_{41}} = \frac{N_{21}}{N_{31}} \times \frac{N_{31}}{N_{41}} = \frac{Z_1 + Z_2}{Z_2} \times \frac{Z_4 Z_5}{Z_4 Z_5 - Z_1 Z'_5}$$

$$\frac{N_{21}}{N_{41}} = \frac{Z_1 Z_4 Z_5 + Z_2 Z_4 Z_5}{Z_2 Z_4 Z_5 - Z_1 Z_2 Z'_5}. \quad \ldots\ldots\ldots\ldots \quad (8)$$

To complete the solution of this part of the problem it is merely a matter of substituting the given numerical values in equation (8), just derived. Hence

$$\frac{N_{41}}{N_{21}} = \frac{Z_2 Z_4 Z_5 - Z_1 Z_2 Z'_5}{Z_1 Z_4 Z_5 + Z_2 Z_4 Z_5}$$

$$= \frac{12 \times 34 \times 46 - 40 \times 12 \times 16}{40 \times 34 \times 46 + 12 \times 34 \times 46}$$

$$= .1363,$$

and therefore if $N_{21} = 2500$ rpm., $N_{41} = 340.75$ rpm.

(b) For this part of the problem the procedure is the same as for the corresponding part of Problem X-34; thus, reasoning as in that problem, we have

$$\frac{T_{14}}{\eta T_{12}} = \frac{N_{21}}{N_{41}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

in which substituting the given numerical values

$$T_{14} = \frac{2500}{340.75} \times 100 \times .9$$

$$= 661.75 \text{ ft. lbs.}$$

Problem X-13.--*Description of the Mechanism.* The mechanism with which the problem deals is partly composed of the set of power gears of the headstock of a metal-turning lathe. The train of gears, Fig. X-13, is a *reverted* unibasal quartic--*reverted* because the center distance for the pair Z_2 and Z_3 is the same as for the pair Z'_3 and Z_4, and thus velocity-axes 12 and 14 can be and have been brought into coincidence with each other. By means of the eccentric device shown in the figure, the operation of which the student

should be sure to understand, gears Z_3 and Z'_3--the back gears they are called--can be thrown into or out of mesh with gears Z_2 and Z_4. When the back gears are in use, stepped pulley 2 is not connected to spindle 4 or gear 4, and power then passes from gear Z_2 to Z_3, from Z_3 to Z'_3, and from Z'_3 to Z_4. When the back gears are thrown out of mesh with the other gears, stepped pulley 2 is then directly connected, by means not shown in the figure, to gear 4, and power from the drive belt passes directly from the pulley to lathe-spindle 4.

Of course the student who depends on this brief and inadequate description to make him acquainted with this very important mechanism is showing deplorably poor judgment and singular lack of enterprise and initiative. What he should do is to go to the shop and examine critically this train of gears on some of the many metal-turning lathes that he will find there.

First Solution of Problem X-13.--What is needed, first of all, is of course the expression in terms of the numbers of teeth for the angular velocity ratio N_{21}/N_{41}. This we get from the two basic ratios of the unibasal quartic, which are

$$\frac{N_{21}}{N_{31}} = -\frac{Z_3}{Z_2} \quad . \quad . \quad . \quad (1), \qquad \frac{N_{31}}{N_{41}} = -\frac{Z_4}{Z'_3} \quad . \quad . \quad . \quad (2)$$

by multiplying the first by the second; thus

$$\frac{N_{21}}{N_{41}} = \frac{N_{21}}{N_{31}} \times \frac{N_{31}}{N_{41}} = -\frac{Z_3}{Z_2} \times -\frac{Z_4}{Z'_3} = \frac{Z_3 Z_4}{Z_2 Z'_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now the problem states (first sentence) that

$$\frac{N_{21}}{N_{41}} = \frac{7}{2} \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

therefore, equation 4,

$$\frac{Z_3 Z_4}{Z_2 Z'_3} = \frac{7}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

And this equation (6) is the formula with which we have to deal in solving the problem.

But before we consider this matter, equation (5) requires a word or two of comment. How did we know, when we wrote down this expression, that N_{21}/N_{41} should be put equal to 7/2 instead of to 2/7? A glance at the sketch gives us the answer, because it shows that 2 is the fast-moving gear and 4 the slow-moving one; and this means that N_{21} is larger than N_{41}, and thus that N_{21}/N_{41} is equal to 7/2 and not to 2/7.

Now we return to equation (6), and our problem with it is almost entirely arithmetic, not kinematic. We have to find four numbers, no one of which is smaller than 15, such that: (1) $Z_3 Z_4 / Z_2 Z'_3 = 7/2$; (2) the center distance is the same for Z_2 and Z_3 as it is for Z'_3 and Z_4, because the train is to be *reverted;* (3) the diametral pitch of Z_2 and Z_3 is 4; (4) the diametral pitch of Z'_3 and Z_4 is 3; and (5) $Z_3/Z_2 = Z_4/Z'_3$ nearly, or with sufficient accuracy.

Condition (2) is expressed symbolically as follows:

$$R_2 + R_3 = R'_3 + R_4 = \textit{Center Distance}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which the R's are the radii of the pitch circles of the four gears. But, Art. 9-14, p. 179, of the text book

$$R_2 = \frac{Z_2}{2P_{23}} \ldots (8), \quad R_3 = \frac{Z_3}{2P_{23}} \ldots (9), \quad R'_3 = \frac{Z'_3}{2P_{34}} \ldots (10)$$

$$R_4 = \frac{Z_4}{2P_{34}} \ldots (11)$$

where P_{23} is the diametral pitch of meshing gears Z_2 and Z_3, and P_{34} is the diametral pitch of meshing gears Z'_3 and Z_4. Thus, substituting from (8), (9), (10), and (11) in (7),

$$\frac{Z_2}{2P_{23}} + \frac{Z_3}{2P_{23}} = \frac{Z'_3}{2P_{34}} + \frac{Z_4}{2P_{34}} \; ; \ldots (12)$$

from which, obviously,

$$\frac{Z_2 + Z_3}{Z'_3 + Z_4} = \frac{P_{23}}{P_{34}} = \frac{4}{3} . \ldots (13)$$

Equation (13) embodies, the student should observe, conditions (2), (3), and (4) of the third paragraph of the solution, this page. We see therefore that the four Zs--Z_2, Z_3, Z'_3, and Z_4--must be so chosen that no one of them is smaller than 15 and so also that equations (6) and (13) are satisfied. Nor must condition (5) of the third paragraph be overlooked. It must be as nearly satisfied as the other conditions, which must be exactly satisfied, will allow. This fifth condition symbolically stated is

$$\frac{Z_3}{Z_2} = \frac{Z_4}{Z'_3} . \ldots (14)$$

From now on the problem is solely one of arithmetic and, to restate it in other terms, is: To so select the four Z's of equation (6) that (1) none of them shall be less than 15, (2) equations (6) and (13) shall be exactly satisfied, and (3) equation (14) shall be satisfied as nearly as possible.

Obviously, if Z_3/Z_2 and Z_4/Z'_3 of equation (6) are to be equal to each other, as equation (14) asserts they should be, then

$$\frac{Z_3}{Z_2} = \frac{Z_4}{Z'_3} = \sqrt{\frac{7}{2}} . \ldots (15)$$

$$= 2 \text{ (with sufficient exactness in many cases).}$$

Then, equation (6),

$$\frac{Z_3}{Z_2} \times \frac{Z_4}{Z'_3} = \frac{2}{1} \times \frac{7}{2 \times 2} ; \ldots (16)$$

and we may write

$$\frac{Z_3}{Z_2} = \frac{2}{1} \ldots (17), \quad \frac{Z_4}{Z'_3} = \frac{7}{4} \ldots (18)$$

Then applying the principle of composition in the *Theory of Proportion* (See p. 148, Hart's College Algebra) to equations (17) and (18) we get

$$\frac{Z_2 + Z_3}{Z_2} = 3 \ldots (19), \quad \frac{Z_3 + Z_4}{Z'_3} = \frac{11}{4} \ldots (20)$$

from which, dividing the first by the second,

$$\frac{Z_2 + Z_3}{Z'_3 + Z_4} \times \frac{Z'_3}{Z_2} = \frac{3}{1} \times \frac{4}{11} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

And now combining equations (13) and (21),

$$\frac{4}{3} \times \frac{Z'_3}{Z_2} = \frac{12}{11},$$

that is,

$$\frac{Z'_3}{Z_2} = \frac{3}{4} \times \frac{12}{11} = \frac{36}{44} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

Thus

$$Z_2 = 44 \text{ and } Z'_3 = 36;$$

and, substituting these numbers in equations (17) and (18), respectively,

$$Z_3 = 88 \text{ and } Z_4 = 63.$$

As a test of the correctness of the four numbers that we have just found, we should see if they satisfy equation (6) and (13). Taking first equation (6):

$$\frac{Z_3}{Z_2} \times \frac{Z_4}{Z'_3} = \frac{88}{44} \times \frac{63}{36}$$

$$= \frac{2}{1} \times \frac{7}{4} = \frac{7}{2}.$$

Taking, next, equation (13)

$$\frac{Z_2 + Z_3}{Z'_3 + Z_4} = \frac{44 + 88}{36 + 63} = \frac{132}{99} = \frac{4}{3}$$

The student should be sure to notice another thing about equation (13). As it is written, it makes the *circular* pitch of the teeth of the gears Z'_3 and Z_4 larger than the circular pitch of gears Z_2 and Z_3; and this means that if the two pinions Z_2 and Z'_3 have the same width of face and are made of the same material, gears Z'_3 and Z_4 will be stronger than gears Z_2 and Z_3, although the two pairs of gears transmit from the belt to the work-spindle of the lathe almost exactly the same amount of power. Z'_3 and Z_4 should be stronger than Z_2 and Z_3 because, their pitch-line speed being smaller than the pitch-line speed of gears 2 and 3, their teeth must be more heavily loaded in order to transmit the same power. This is one of the fundamental principles of machine design, a subject of which mechanism is a part.

Second Solution of Problem X-13.--If in satisfying equation (15) of the preceding solution we take not 2 but some other value as an approximation to the square root of 7/2, we get other values for the four Z's. If we take, for instance, 3/2 for a sufficiently close approximation to the square root of 7/2 and operate with it just as we did with the value 2, we get the following set of numbers

$$Z_2 = 32, \quad Z_3 = 48, \quad Z'_3 = 18, \quad Z_4 = 42,$$

a very much better set than that of Solution 1.

The student should compare the two sets of numbers and consider why and in what way the second set is better than the first one. He should be certain, moreover, to try to find other sets of numbers that will meet in a more practical manner the five conditions specified in the third paragraph of the first solution.

Problem X-14.--To solve this problem, which also deals with the head-stock of a metal-turning lathe, we use the same figure, the same procedure, and the same basic formulas that we used in solving Problem X-13.

Thus, equation (6), of Problem X-13,

$$\frac{N_{21}}{N_{41}} = \frac{Z_3}{Z_2} \times \frac{Z_4}{Z'_3} = \frac{19}{1}$$

$$= \frac{4}{1} \times \frac{19}{1} \times \frac{1}{4}. \quad \ldots\ldots\ldots\ldots \quad (1)$$

if we take 4 as a sufficiently close approximation to the square root of 19, which, to four places of decimals, is 4.3589+. Then

$$\frac{Z_3}{Z_2} = 4 \ldots (2), \qquad \frac{Z_4}{Z'_3} = \frac{19}{4} \ldots (3)$$

$$\frac{Z_2 + Z_3}{Z_2} = 5 \ldots (4), \qquad \frac{Z'_3 + Z_4}{Z'_3} = \frac{23}{4} \ldots (5)$$

$$\frac{Z_2 + Z_3}{Z'_3 + Z_4} \times \frac{Z'_3}{Z_2} = 5 \times \frac{4}{23} = \frac{20}{23}. \quad \ldots\ldots\ldots\ldots \quad (6)$$

In (6) we must take $(Z_2 + Z_3)/(Z'_3 + Z_4)$ equal to 1 because the problem states that $P_{23} = P_{34}$. (See equation (13) of Problem X-13.) From equation (6), putting $(Z_2 + Z_3)/(Z'_3 \; Z_4)$ equal to 1, we get

$$\frac{Z'_3}{Z_2} = \frac{20}{23}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Thus

$$Z_2 = 23 \text{ and } Z'_3 = 20,$$

and, equation (2) and (3),

$$Z_3 = 92 \text{ and } Z_4 = 95.$$

Moreover

$$Z_2 + Z_3 = 23 + 92 = 115.$$

and

$$Z'_3 + Z_4 = 20 + 95 = 115.$$

Two of these numbers are larger, however, than the statement of the problem allows, so that we must make a second trial.

Second Solution of Problem X-14.--If we take 5 as a sufficiently close approximation to the square root of 19, equation (1), and proceed just as we did in making the first solution, we obtain the following set of numbers:

$$Z_2 = 12, \quad Z_3 = 60, \quad Z'_3 = 15, \quad Z_4 = 57.$$

But neither of these solutions is, the student should be sure to observe, very practical, because of the condition $P_{23} = P_{34}$. The slow-moving gears, since they are the more heavily loaded, should have the stronger teeth; that is, P_{23} (for the fast-moving gears) should be larger than P_{34} (for the slow-moving gears). Let us therefore make a third solution.

Third Solution of the Problem.--Let us assume for this solution that 5 is a sufficiently close approximation to the square root of 19 and, to overcome one of the defects of the other solutions, that $P_{23}/P_{34} = 4/3$. Then equations (1), (2), (3), (4), (5), (6), operated with as usual, give us

$$\frac{Z_3}{Z_2} \times \frac{Z_4}{Z'_3} = \frac{5}{1} \times \frac{19}{1} \times \frac{1}{5}. \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{Z_3}{Z_2} = \frac{5}{1} \ldots (2), \qquad \frac{Z_4}{Z_3} = \frac{19}{5} \ldots (3)$$

$$\frac{Z_2 + Z_3}{Z_2} = 6 \ldots (4), \qquad \frac{Z'_3 + Z_4}{Z'_3} = \frac{24}{5} \ldots (5)$$

$$\frac{Z_2 + Z_3}{Z'_3 + Z_4} \times \frac{Z'_3}{Z_2} = 6 \times \frac{5}{24} = \frac{5}{4}. \quad \ldots\ldots\ldots\ldots \quad (6)$$

Then from (6), substituting for $(Z_2 + Z_3)/(Z'_3 + Z_4)$, $4/3$, we get

$$\frac{Z'_3}{Z_2} = \frac{3}{4} \times \frac{5}{4} = \frac{15}{16}, \quad \ldots\ldots\ldots\ldots \quad (7)$$

that is, $$Z_2 = 16, \quad Z'_3 = 15;$$

and therefore, equations (2) and (3),

$$Z_3 = 80, \quad Z_4 = 57.$$

One of these numbers is, it is true, larger than the problem stipulates, but it would be perfectly acceptable in any practical application.

For these four numbers of teeth

$$\frac{Z_2 + Z_3}{Z'_3 + Z_4} = \frac{16 + 80}{15 + 57} = \frac{96}{72} = \frac{4}{3},$$

and

$$\frac{Z_3}{Z_2} \times \frac{Z_4}{Z'_3} = \frac{80}{16} \times \frac{57}{15} = \frac{5}{1} \times \frac{19}{5}$$

as required.

Problem X-15.--This problem is concerned with a spur-gear speed-changing device or variator, such as is used so extensively in automobile practice. The mechanism, Fig. X-15, is composed of three reverted unibasal quartic spur-gear trains, equal gears Z_2 and Z_3 being common to each of these quartics. Shaft 2 runs at the constant speed of 100 rpm., and shaft 4 is driven at one or the other of the three speeds 20, 200, or 860 rpm., according as gear-pairs Z'_3 and Z'_4 or Z''_3 and Z''_4 or Z'''_3 and Z'''_4, respectively, are engaged. Gears Z'_4, Z''_4, and Z'''_4 are rigidly fastened together to form a triplex gear that is

splined to shaft 4. By suitable mechanism, not shown in the figure, this triplex gear can be slid along shaft 4 so that a particular one of its three component gears can be brought into mesh with a particular one of the three gears keyed rigidly to shaft 3.

Solution of Problem X-15.--By the methods fully explained in the solutions of the two preceding problems we have to design three reverted unibasal quartics. (a) We start with the quartic composed of the two cubics 1 Z_2 Z_3 and 1 Z_3'' Z_4''. For this we have, equation (4), Problem X-13, Solution 1,

$$\frac{N_{21}}{N_{41}} = \frac{Z_3 Z_4''}{Z_2 Z_3''} = \frac{100}{20} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

But by the conditions of the problem

$$Z_2 = Z_3 \text{ and } Z_4'' = 80;$$

therefore, equation (1),

$$Z_3'' = 16,$$

and

$$Z_3'' + Z_4'' = 16 + 80 = 96.$$

Now the center-distance for *equal* gears Z_2 and Z_3 must be the same as for gears Z_3'' and Z_4'' and by the conditions of the problem the diametral pitch of Z_2 and Z_3 must be the same as for Z_3'' and Z_4''; therefore

$$Z_3'' + Z_4'' = 96 = Z_2 + Z_3. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Again, the conditions of the problem give

$$Z_2 = Z_3;$$

therefore

$$Z_2 = Z_3 = 48.$$

(b) Again, taking the second quartic and modifying suitably the symbols of equation (1) above, we have

$$\frac{N_{21}}{N_{41}} = \frac{Z_3 Z_4'''}{Z_2 Z_3'''} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

from which, since for this case shaft 4 is to run at a speed of 200 rpm., we have

$$\frac{Z_3 Z_4'''}{Z_2 Z_3'''} = \frac{100}{200} = \frac{1}{2}.$$

But, as in case (a),

$$Z_2 = Z_3;$$

therefore

$$Z_3''' = 2Z_4'''; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

moreover,

$$Z_3''' + Z_4''' = 96, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

because the center distance and diametral pitch for this pair of gears must be the same as for the pair Z_2 and Z_3. From (4) and (5) we find

$$Z_3''' = 64, \quad Z_4''' = 32.$$

(c) Taking the third quartic and modifying again the symbols of equation (1), we get

$$\frac{N_{21}}{N_{41}} = \frac{Z_3 Z_4'}{Z_2 Z_3'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

But for this case

$$N_{41} = 860 \text{ rpm.},$$

and, as for cases (a) and (b),

$$Z_2 = Z_3;$$

moreover, for the reasons already twice stated,

$$Z_3' + Z_4' = 96; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

therefore from (6) above

$$\frac{Z_4'}{Z_3'} = \frac{100}{860} = \frac{5}{43}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

and from (7) and (8)

$$Z_3' = 86, \quad Z_4' = 10.$$

Problem X-16.--The mechanism of this problem, Fig. X-16, is also a speed-varying device--one much used on machine tools. It is a special unibasal quintic, composed of three separate cubics: $1\ Z_2\ Z_3$, $1\ Z_2'\ Z_4$, and $1\ Z_2''\ Z_5$. These cubics have links 1 and 2 in common, and link 2 is a triplex gear that meshes each of the other gears. By means--friction clutches perhaps--not shown in the figure any one of the gears Z_3, Z_4, Z_5 can be connected to shaft 345 and the other two disconnected; so that shaft 345 can be driven at three different speeds by triplex gear 2.

Solution of Problem X-16.--Take, first, the cubic whose gears are Z_2 and Z_3. For this, not bothering about the sign, we have

$$\frac{N_{21}}{N_{31}} = \frac{Z_3}{Z_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now the conditions of the problem give

$$N_{21} = 1$$

and

$$N_{31} = 2.5;$$

so that

$$\frac{Z_3}{Z_2} = \frac{1}{2.5} = \frac{2}{5}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Taking, next, the cubic whose gears are Z_2' and Z_4 and suitably altering the symbols of equation (1), we have

$$\frac{N_{21}}{N_{41}} = \frac{Z_4}{Z_2'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

For this case

$$N_{21} = 1,$$

$$N_{41} = \frac{5}{4};$$

therefore, equation (3),

$$\frac{Z_4}{Z_2'} = \frac{4}{5}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Proceeding in exactly the same way for the third cubic, we obtain

$$\frac{Z_5}{Z_4} = 2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

And then from equations (2), (4), and (5) by the theory of proportion

$$\frac{Z_2 + Z_3}{Z_2} = \frac{7}{5}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

$$\frac{Z_2' + Z_4}{Z_2'} = \frac{9}{5}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

$$\frac{Z_2'' + Z_5}{Z_2''} = \frac{3}{1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

And now dividing (6) by (7)

$$\frac{Z_2 + Z_3}{Z_2' + Z_4} \times \frac{Z_2'}{Z_2} = \frac{7}{5} \times \frac{5}{9} = \frac{7}{9}; \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

but since $(Z_2 + Z_3)/(Z_2' + Z_4) = 1$, (why?)

$$\frac{Z_2'}{Z_2} = \frac{7}{9}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

and then dividing (6) by (8):

$$\frac{Z_2 + Z_3}{Z_2'' + Z_5} \times \frac{Z_2''}{Z_2} = \frac{7}{5} \times \frac{1}{3} = \frac{7}{15}; \quad \ldots\ldots\ldots\ldots \quad (11)$$

but, again, since $(Z_2 + Z_3)/(Z_2'' + Z_5) = 1$, (why?)

$$\frac{Z_2''}{Z_2} = \frac{7}{15}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

From equations (2), (10), and (12) we see that Z_2 must be an integral multiple of 5, 9, and 15, for otherwise Z_3, Z_2, and Z_2'' would not be integers. Why? From equations (4)

and (10) we see that Z_2' must be an integral multiple of 5 and 7. Integers Z_2 and Z_2' must, moreover, bear to each other the ratio given by equation (10). Thus, equations (2), (10), and (12),

$$\frac{Z_3}{Z_2} = \frac{2}{5} \times \frac{9}{9} = \frac{18}{45}, \quad \ldots \ldots \ldots \ldots \ldots \quad (13)$$

$$\frac{Z_2}{Z_2} = \frac{7}{9} \times \frac{5}{5} = \frac{35}{45}, \quad \ldots \ldots \ldots \ldots \ldots \quad (14)$$

$$\frac{Z_2'}{Z_2} = \frac{7}{15} \times \frac{3}{3} = \frac{21}{45}, \quad \ldots \ldots \ldots \ldots \ldots \quad (15)$$

And, similarly, from equations (4) and (5)

$$\frac{Z_4}{Z_2'} = \frac{4}{5} \times \frac{7}{7} = \frac{28}{35}, \quad \ldots \ldots \ldots \ldots \ldots \quad (16)$$

$$\frac{Z_5}{Z_2''} = \frac{2}{1} \times \frac{21}{21} = \frac{42}{21}. \quad \ldots \ldots \ldots \ldots \ldots \quad (17)$$

So that the required six numbers of teeth are:

$$Z_2 = 45, \quad Z_3' = 35, \quad Z_2'' = 21, \quad Z_3 = 18, \quad Z_4 = 28, \quad Z_5 = 42.$$

The student should be sure to test these numbers to see if the center distance for each pair of gears is the same.

Problem X-17.--Description of the Mechanism. The mechanism, Fig. X-17, with which we are concerned in this problem is another speed-changing device. We may call it a variable unibasal quartic because in it the angular velocity ratio N_{21}/N_{41} can be given three different values by moving triplex gear 3 to three different positions on the shaft to which it is splined as shown in the figure. So that if shaft 2 runs at a constant speed, shaft 4 will turn with one or another of three different speeds, depending upon the position of triplex gear 3.

Solution.--(a) Since, according to the conditions of the problem, N_{21} and N_{41} must be equal to each other when Z_2 is driving Z_4 through idler Z_3, Z_2 must be equal to Z_4.

From the two cubics

$$\frac{N_{21}}{N_{31}} = -\frac{Z_3'}{Z_2'} \ldots (1), \qquad \frac{N_{31}}{N_{41}} = -\frac{Z_4}{Z_3} \ldots (2)$$

of the quartic spur-gear train $1Z_2'Z_3'Z_3Z_4$ we have

$$\frac{N_{21}}{N_{41}} = \frac{N_{21}}{N_{31}} \times \frac{N_{31}}{N_{41}} = \frac{Z_3'Z_4}{Z_2'Z_3} \ldots (3)$$

But $N_{21} = 60$ *rpm.*, and for this first quartic $N_{41} = 240$ *rpm.*; accordingly

$$\frac{Z_3'Z_4}{Z_2'Z_3} = \frac{60}{240} = \frac{1}{4}. \quad \ldots \ldots \ldots \ldots \ldots \quad (4)$$

$$= \frac{1}{2} \times \frac{1}{2},$$

if we divide the ratio for the quartic equally between the two cubics of which it is composed.

Therefore

$$\frac{Z'_3}{Z'_2} = \frac{1}{2} \ldots (5), \quad \frac{Z_4}{Z_3} = \frac{1}{2} \ldots (6)$$

from which, since center distances 12-13 and 13-14 are equal and all of the gears have the same pitch, it follows that

$$Z'_2 = Z_3 \ldots (7) \text{ and } Z'_3 = Z_4 \ldots (8)$$

Now, as we have already seen, $Z_2 = Z_4$; thus, equation (8), $Z'_3 = Z_2$.

(b) For the quartic $1Z_2Z_3Z''_3Z'_4$ for which $N_{41} = 15$ rpm., we have, similarly,

$$\frac{N_{21}}{N_{41}} = \frac{Z_3Z'_4}{Z_2Z''_3} = \frac{60}{15} = 4 \ldots (9)$$

$$= \frac{2}{1} \times \frac{2}{1},$$

if we divide the ratio of the quartic equally between the two cubics. Thus

$$\frac{Z_3}{Z_2} = 2 \ldots (10), \quad \frac{Z'_4}{Z''_3} = 2 \ldots (11)$$

which, since here also the center distances are equal and the pitch is the same for all of the gears,

$$Z_2 = Z''_3 \ldots (12) \text{ and } Z_3 = Z'_4 \ldots (13);$$

moreover,

$$Z'_2 = Z_3 = Z'_4. \ldots (14)$$

and

$$Z_2 = Z'_3 = Z''_3 = Z_4. \ldots (15)$$

(c) It is necessary, therefore, in view of equations (14) and (15), to determine only Z_2 and Z_3 so that the gears to which they belong will have the given center distance, 13.5 inches. This we do as follows, using equation (10) and the equation

$$R_3 + R_2 = 13.5, \ldots (16)$$

which expresses the obvious fact that the center distance is equal to the sum of the radii of the pitch circles. Now from elementary considerations concerning spur-gears

$$R_3 = \frac{Z_3}{2P_{23}} \text{ and } R_2 = \frac{Z_2}{2P_{23}};$$

therefore, substituting in (16),

$$Z_3 + Z_2 = 13.5 \times 2P_{23} = 13.5 \times 2 \times 4, \ldots (17)$$

for factor 4 being the given diametral pitch.

From equations (10) and (17)

$$Z_2 = 36 \text{ and } Z_3 = 72$$

and

$$Z''_3 = Z'_3 = Z_4 = 36.$$

Problem XII-16.--In this problem we are concerned with the very important worm-and-wheel mechanism, one of the oldest, one of the most useful, and one of the most interesting of mechanisms. Egyptian and Chaldean engineers were probably familiar with its crude forms three or four thousand years ago. It is a *screw* mechanism, the worm and wheel having screw, not complanar, motion relative to each other, the most general kind of relative motion.

It is now possible to design such a mechanism so scientifically that it will have an efficiency as high as 97 or 98%. Forty years ago, however, engineers could get out of it an efficiency not much higher than 45%; consequently it was seldom used prior to that time as a power drive.

The exact nature of the contact between the threads of the worm and the teeth of the wheel in the Hindley worm and wheel is not fully understood. By some it is believed to be surface contact, by others line contact. In the ordinary worm and wheel, such, for example as the one on the Brown and Sharpe model, in which an axial section of the worm is the same as the right section of an involute rack, the contact is quite well understood and is single- or double- or triple-*line* contact, according as one or two or three pairs of teeth of the wheel are engaged.

If the worm is single-threaded the angular velocity ratio--the ratio, that is, of the number of turns of the worm to the number of turns of the wheel in the same time--is equal to the number of teeth on the wheel; if, however, the worm is double-threaded or triple-threaded or n-threaded, the angular velocity ratio is one-half or one-third or 1/nth the number of teeth on the wheel, respectively. If, to put the statement symbolically, N_{21} is the number of turns of the worm in a minute, N_{31} the number of turns of the wheel in the same time, Z_2 the number of threads on the worm, and Z_3 the number of teeth on the wheel, then, paying no attention to signs,

$$\frac{N_{21}}{N_{31}} = \frac{Z_3}{Z_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The *lead* of a screw or worm is the distance that the thread advances parallel to the axis of the screw for each turn of the thread about the axis. The *pitch* of a screw is the axial distance from a point on one thread to the corresponding point on the adjacent thread. For a single-threaded screw *lead* and *pitch* are the same; for a double- or triple- or n-threaded screw the pitch is one-half or one-third or 1/nth the lead. In other words, the lead in inches is equal to the pitch in inches multiplied by the number of threads on the screw. Symbolically stated,

$$L_{23} = Z_2 P'_{23}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

in which L_{23}, Z_2, and P'_{23} are, respectively, the lead, the number of threads, and the pitch.

Solution of Problem XII-16.--(a) The vertical distance in inches that a point on the rope or weight travels in a minute is obviously $2\pi R'_3 N_{31}$, in which R'_3 is the radius of the rope-drum and N_{31} the number of turns that the drum makes in a minute. If therefore, we call this distance d, we have evidently,

$$d = 2\pi R'_3 N_{31}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

But, equation (1),

$$\frac{N_{31}}{N_{21}} = \frac{Z_2}{Z_3}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

therefore

$$d = 2\pi R' \frac{Z_2}{Z_3} N_{21},$$

or, solving for the required quantity,

$$Z_3 = \frac{2\pi R'_3 Z_2}{d} N_{21} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

$$= \frac{2\pi \times 7 \times 3}{20} \times 30$$

$$= 197.92+.$$

Now since 198 is the nearest integer we take that as the answer.

Because, however, 198 is not the exact number of teeth that the conditions of the problem require, the weight will not move exactly 20 inches when the worm turns 30 times. How far *will* it move?

(b) To solve the second part of the problem we use the principle of work. The work put into the machine at the crank in a minute is clearly $2\pi R_2 N_{21} F_2$, where R_2 in inches is the length of the crank, F_2 the force in pounds acting at the end of this crank, and N_{21} the number of turns per minute that the crank makes. The work done in a minute in lifting the weight is, moreover, clearly $2\pi R'_3 N_{31} W_3$, where R'_3 in inches is the radius of the rope drum, N_{31} the number of turns per minute of the rope drum, and W_3 the weight in pounds suspended from the rope. Now in conformity with the *Law of the Conservation of Energy* these two quantities of work would be equal if none of the energy put into the machine at the crank were wasted in friction. But 40% of it is so wasted, the problem says; so that only 60% of the energy put into the machine at the crank goes to the performance of useful work in lifting the weight; in other words,

$$.6 \times 2\pi R_2 N_{21} F_2 = 2\pi R'_3 N_{31} W_3$$

or

$$F_2 = \frac{R'_3 W_3 N_{31}}{.6 R_2 N_{21}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

from which, using equation (1), we get

$$F_2 = \frac{R'_3 W_3 Z_2}{.6 R_2 Z_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

$$= \frac{7 \times 8800 \times 3}{.6 \times 16.66 \times 198}$$

$$= 93 \text{ lbs., approx.}$$

(c) Let the student imagine himself standing on the right-hand side of the mechanism, looking at it from right to left. And let the crank turn counterclockwise as the problem

specifies. Now because the weight is being lifted, worm-wheel 3 must be turning clockwise, looked at from in front of the machine. Hence the top of the worm-wheel 3 must be moving from left to right; but for worm 2 to cause the top of the wheel to move thus when crank 2 is turned counterclockwise, it (the worm) must have a left-handed thread.

(The student who has trouble in seeing the truth of this conclusion may be sure that what he needs to do is to gain greater familiarity with worm-and-wheel models in the lecture room.)

Problem XII-17.--Here we have to deal with a mechanism or machine that is composed of two worm-and-wheel drives. How the device operates is perfectly evident from the figure.

Solution of Problem XII-17.--Applying the principle of work, and noting the symbols on the figure and the conditions specified in the statement of the problem, we may at once write

$$.4\times 2\pi R_4 N_{41} F_4 = 2\pi R'_3 N_{31} W_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where .4 is the coefficient of efficiency. From (1) we have

$$\frac{N_{31}}{N_{41}} = \frac{.4\times R_4 F_4}{R'_3 W_3} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But for the first worm and wheel

$$\frac{N_{31}}{N_{21}} = \frac{Z_2}{Z_3} , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for the second

$$\frac{N_{21}}{N_{41}} = \frac{Z_4}{Z'_2} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

thus, combining (3) and (4),

$$\frac{N_{31}}{N_{41}} = \frac{Z_2 Z_4}{Z_3 Z'_2} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (2) and (5), therefore

$$\frac{Z_2 Z_4}{Z_3 Z'_2} = \frac{.4 R_4 F_4}{R'_3 W_3} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In equation (6) appear two unknown quantities, Z_4 and Z'_2; moreover, the quantity that we want to find--the diameter D'_2 of worm wheel Z'_2--does not appear in it. Thus the problem cannot be solved with this equation as it stands. But from elementary considerations we have at our disposal

$$P'_{24} Z'_2 = \pi D'_2 , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which P'_{24} is the circular pitch of worm wheel Z'_2; therefore, equations (6) and (7),

$$\frac{Z_2 Z_4 P'_{24}}{\pi D'_2 Z_3} = \frac{.4 R_4 F_4}{R'_3 W_3} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Moreover, when a worm and wheel mesh each other the circular pitch of the worm wheel must of course be equal to the linear pitch of the worm; thus, remembering what was said about the relation between the pitch and the lead of a screw or worm in the preceding problem we see that

$$Z_4 P'_{24} = L_{24} . \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

Therefore equation (8) becomes

$$\frac{Z_2 L_{24}}{\pi D'_2 Z_3} = \frac{.4 R_4 F_4}{R'_3 W_3}, \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

from which, solving for D'_2, we have

$$D'_2 = \frac{Z_2 W_3 L_{24} R'_3}{.4 \times \pi R_4 Z_3 F_4}. \quad \ldots\ldots\ldots\ldots\ldots \quad (11)$$

$$= \frac{2 \times 25344 \times 5/8 \times 6}{.4 \times \pi \times 16 \times 36 \times 20}$$

$$= 13.12''.$$

Problem XII-18.--Here we have to deal with a train of gears that is composed of two worm-and-wheel mechanisms and a cubic spur-gear mechanism, the latter connecting the two former. The figure makes perfectly clear how the device works.

Solution of Problem XII-18.--(a) This part of the problem is concerned with the relative motions of the three links, 1, 2, 5, because the question is:

How many clockwise (see the arrowhead) turns of crank 2, relative to 1 are necessary to cause 2 to turn *once* relative to 5?

I specify in the question *one* turn and not some other whole number of turns of 2 relative to 5, because after one turn of one of these pieces relative to the other they will be again, with the least amount of motion, in the same relative position that they occupied before motion began.

That 2 and 5 turn in the same sense relative to 1 is easily shown as follows:

In distinguishing clockwise from counterclockwise motion of any one of the links 1, 2, 5 relative to any other one of these three links, the observer is supposed to stand on the left of the mechanism and look at it; in distinguishing clockwise from counterclockwise motion of any one of the links 1, 3, 4 relative to any other one of these three links, the observer is supposed to stand in front of the machine and look at it.

If now worm 2 is turned clockwise as the problem requires, the *top* of worm wheel 3 will move towards the left of the observer standing in front of the machine, because worm 2 is right-handed. But if the top of this wheel moves towards the left the wheel turns counterclockwise; and if gear 3 turns counterclockwise gear 4 turns clockwise; and if gear 4 (or worm 4) turns clockwise, the bottom of worm wheel 5 will move away from the observer stationed in front of the machine, because worm 4 is left-handed. But if the *bottom* of worm wheel 5 moves thus the wheel evidently turns clockwise as seen by the observer on the left. Hence the statement that 2 and 5 turn in the same sense relative to 1 is correct.

The student will now observe that axes 12, 15, and 25 coincide with each other permanently, and hence that links 1, 2, and 5 have complanar motion relative to each other, and for their relative motions, therefore, all the laws of complanar motion hold. Thus, Theorem IX,

$$N_{21} = N_{51} + N_{25},$$

or, multiplying through by t to convert turns per minute to turns,

$$N_{21} t = N_{51} t + N_{25} t. \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now noting the symbols on the figure and for the moment disregarding signs, we have for the unibasal quintic the basic ratios

$$\frac{N_{21}t}{N_{31}t} = \frac{Z_3}{Z_2} \quad . \quad . \quad . \quad (2), \qquad \frac{N_{31}t}{N_{41}t} = \frac{D'_4}{D'_3} \quad . \quad . \quad . \quad (3), \qquad \frac{N_{41}t}{N_{51}t} = \frac{Z_5}{Z_4} \quad . \quad . \quad . \quad (4)$$

from which, by multiplication,

$$\frac{N_{21}t}{N_{51}t} = \frac{N_{21}t}{N_{31}t} \times \frac{N_{31}t}{N_{41}t} \times \frac{N_{41}t}{N_{31}t} = \frac{Z_3}{Z_2} \times \frac{Z'_4}{D'_3} \times \frac{Z_5}{Z_4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the sign of which is positive because, as we have just seen, 2 and 5 turn in the same sense relative to 1. From (5) eliminating Z_5, which is not known, with the equation

$$Z_5 P'_{45} = \pi D_5,$$

we have

$$\frac{N_{21}t}{N_{51}t} = \frac{Z_3}{Z_2} \times \frac{D'_4}{D'_3} \times \frac{\pi D_5}{P'_{45} Z_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But, as in the two preceding problems,

$$Z_4 P'_{45} = L_{45};$$

therefore

$$\frac{N_{21}t}{N_{51}t} = \frac{Z_3}{Z_2} \times \frac{D'_4}{D'_3} \times \frac{\pi D_5}{L_{45}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which everything on the right-hand side is known. Now with (7) eliminating N_{51} from equation (1) we get

$$N_{21}t = \frac{\pi Z_3 D'_4 D_5}{\pi Z_3 D'_4 D_5 - Z_2 D'_3 L_{45}} (N_{25}t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (8) we see that, since the second term in the denominator is numerically less than the first term, the sign of $N_{25}t$ must be the same as the sign of $N_{21}t$. In other words, we see that 2 must turn in the same sense relative to 5 that it does relative to 1, that is, clockwise. We must, therefore, put for $N_{25}t$ in formula (25) the value +1 to get an equation with which to calculate $N_{21}t$.

The arithmetic of the problem is left to the student.

(b) If worm 4 is right-handed, not left-handed as so far supposed, then the sign of the right-hand side of equation (7) will be negative and the sign between the two terms in the denominator of equation (8) will be plus. Thus we may write equation (8) with the double sign between the terms in the denominator:

$$N_{21}t = \frac{\pi Z_3 D'_4 D_5}{\pi Z_3 D'_4 D_5 \pm Z_2 D'_3 L_{45}} (N_{25}t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

wherein the plus or the minus sign is to be taken according as worm 4 is right- or left-handed.

With equation (9), then, either of the first two questions of this problem can be answered. And for both questions $(N_{25}t)$ is unity and has the same sign as $(N_{21}t)$

This problem is an excellent and a most interesting application of mathematical signs, one of the most difficult questions in elementary mathematics, and the student ought to be

sure to give the problem careful study from this point of view.

(c) For the solution of this part of the problem we must appeal again to the principle of work. Noting the symbols on the figure and writing the equation of work on the supposition--very far from the truth--that the efficiency of the machine is 100%, we have

$$2\pi R_2(N_{21}t)F_2 = 2\pi R'_5(N_{51}t)W_5,$$

from which

$$F_2 = \frac{R'_5 W_5}{R_2}\frac{(N_{51}t)}{(N_{21}t)}. \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

And then, eliminating $(N_{51}t)/(N_{21}t)$ with equation (7), we obtain

$$F_2 = \frac{R'_5}{R_2} \times \frac{Z_2}{Z_3} \times \frac{D'_3}{D'_4} \times \frac{L_{45}}{\pi D_5} W_5 \quad \ldots\ldots\ldots\ldots \quad (11)$$

with which to compute F_2, the force required.

The student ought to test all such equations as (11) for consistency of units. It is very easy to do. On the right-hand side of the equation there are four ratios multiplying W_5, a force. The first R_5/R_2, is the ratio of two linear quantities, both expressed, of course, in the same unit; therefore R'_5/R_2 is an abstract number. Z_2 and Z_3 are both abstract numbers. D'_3/D'_4 is the ratio of two linear quantities, both expressed in the same unit; therefore D'_3/D'_4 is an abstract number. L_{45}/D_5 is the ratio of two linear quantities both expressed in the same unit; therefore L_{45}/D_5 is an abstract number. Thus on the right-hand side of the equation we have a force multiplied by an abstract number; and on the left-hand side a force. Therefore the equation is consistent.

Supplementary Remarks on Problem XII-18.--The statement of the problem in the text book gives the efficiency of the machine as 100%, but of course no such efficiency as this can be realized in practice. With both worms either single- or double-threaded the efficiency of such a train is likely to be a good deal below 50 per cent. If, however, the worm-and-wheel mechanisms are carefully designed with a view to their yielding high efficiencies, it is possible to get the efficiency of the machine up to 90 per cent or even somewhat higher. Let us carefully consider what would have to be done to insure such an efficiency.

In the first place, the two worms instead of being single- or double-threaded would have to be quintuple- or sextuple-threaded, because the efficiency of a worm-and-wheel mechanism increases with the angle of helix of the worm thread up to a value for this angle of nearly 45% and then begins to decrease. But a large thread angle can be obtained in a practical manner only with multiple-threaded worms. In the second place, it would probably be necessary to use ball- or roller-bearings to support the three shafts.

If, however, worm 2 were sextuple-threaded instead of double-threaded, the number of teeth on worm wheel 3 would have to be increased from 32 to 96 to keep the angular velocity ratio N_{21}/N_{31} unchanged in value; because, equation (2),

$$\frac{N_{21}}{N_{31}} = \frac{Z_3}{Z_2} = \frac{32}{2} = \frac{32 \times 3}{2 \times 3} = \frac{96}{6}. \quad \ldots\ldots\ldots\ldots \quad (12)$$

And for a diametral pitch of 5 the diameter of the pitch circle of wheel 3 would have to be

$$D_3 = \frac{Z_3}{P_{23}} = \frac{96}{5} = 19.2''. \quad \ldots\ldots\ldots\ldots \quad (13)$$

The number of teeth on worm wheel 5 if N_{41}/N_{51} is to remain unchanged when worm 4 is made sextuple-threaded is found with the aid of equation (4). From which and the relation

$$Z_5 P'_{45} = \pi D_5 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

we have

$$\frac{N_{41}}{N_{51}} = \frac{Z_5}{Z_4} = \frac{\pi D_5}{Z_4 P'_{45}} = \frac{\pi D_5}{L_{45}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

in which L_{45} is the given lead of worm 4. From (15) and the given data of the problem

$$\frac{N_{41}}{N_{51}} = \frac{\pi D_5}{L_{45}} = \frac{\pi \times 10.83}{2} = \frac{34.023}{2}$$

$$= \frac{34.023}{2} \times \frac{3}{3} = \frac{102.069}{6}.$$

That is, the number of teeth on worm wheel 5 would have to be 102. And if for this wheel we choose a diametral pitch of 4, we get for its pitch diameter, D_5, the value of 25.5".

Efficiency of the Machine.--Perhaps the student would be interested in seeing how the efficiency of the whole machine is related to the efficiencies of its component mechanisms. Let us, therefore, work out this relation. The reasoning is very simple and easily followed.

Let:

1. η_{23} = The efficiency of the component consisting of baselink 1, worm 2, and worm-wheel 3;
2. η_{34} = The efficiency of the component consisting of baselink 1 and spur-gears 3 and 4;
3. η_{45} = The efficiency of the component consisting of baselink 1, worm 4, and worm-wheel 5.

Then by the principle of work, applied in turn to each of the component mechanisms, we have

$$\frac{T_{12}}{T_{13}} = \frac{1}{\eta_{23}} \times \frac{N_{31}}{N_{21}} = \frac{1}{\eta_{23}} \times \frac{Z_2}{Z_3}, \quad \ldots\ldots\ldots\ldots \quad (16)$$

$$\frac{T_{13}}{T_{14}} = \frac{1}{\eta_{34}} \times \frac{N_{41}}{N_{31}} = \frac{1}{\eta_{34}} \times \frac{D'_3}{D'_4}, \quad \ldots\ldots\ldots\ldots \quad (17)$$

$$\frac{T_{14}}{T_{15}} = \frac{1}{\eta_{45}} \times \frac{N_{51}}{N_{41}} = \frac{1}{\eta_{45}} \times \frac{Z_4}{Z_5}. \quad \ldots\ldots\ldots\ldots \quad (18)$$

Therefore, by multiplication,

$$\frac{T_{12}}{T_{15}} = \frac{T_{12}}{T_{13}} \times \frac{T_{13}}{T_{14}} \times \frac{T_{14}}{T_{15}} = \frac{1}{\eta_{23}} \times \frac{1}{\eta_{34}} \times \frac{1}{\eta_{45}} \times \frac{Z_2}{Z_3} \times \frac{D'_3}{D'_4} \times \frac{Z_4}{Z_5}. \quad \ldots\ldots \quad (19)$$

But we know also that

$$\frac{T_{12}}{T_{15}} = \frac{1}{\eta_{25}} \frac{N_{51}}{N_{21}} = \frac{1}{\eta_{25}} \frac{Z_2}{Z_3} \times \frac{D'_3}{D'_4} \times \frac{Z_4}{Z_5}, \quad \ldots\ldots\ldots\ldots \quad (20)$$

where η_{25} is the efficiency of the whole machine; accordingly, equations (19) and (20),

$$\frac{1}{\eta_{25}} = \frac{1}{\eta_{23}} \times \frac{1}{\eta_{34}} \times \frac{1}{\eta_{45}},$$

or

$$\eta_{25} = \eta_{23}\,\eta_{34}\,\eta_{45}. \quad \ldots\ldots\ldots\ldots \quad (21)$$

That is to say, the efficiency of the whole machine is equal to the product of the efficiencies of its component mechanisms.

This, moreover, it is now obvious, is a general not a special principle; a principle, therefore, applicable to all machines.

Because the designer would have to know them, the student ought to be interested in learning how to calculate: (1) the pressure between the threads of worm 2 and the teeth of worm-wheel 3; (2) the pressure between the teeth of spur-gears 3 and 4; and (3) the pressure between the threads of worm 4 and the teeth of worm-wheel 5.

The Pressure between the Threads of Worm 2 and the Teeth of Worm-Wheel 3.--To calculate this force we must use equation (16); from which we have

$$T_{13} = \eta_{23} T_{12} \frac{Z_3}{Z_2}, \quad \ldots\ldots\ldots\ldots \quad (22)$$

in which T_{12} is known from equation (20) and η_{23} is known from experiment or experience with such machines. But also, taking moments about axis 13,

$$T_{13} = R_3 F_{23},$$

or

$$F_{23} = \frac{T_{13}}{R_3}, \quad \ldots\ldots\ldots\ldots \quad (23)$$

in which R_3 is the radius of the pitch circle of worm-wheel 3 and F_{23} the component of the pressure between the worm and wheel that acts tangential to this pitch circle.

Pressure between the Teeth of the Spur Gears.--For this force we have, similarly,

$$T_{13} = R'_3 F_{34}$$

or

$$F_{34} = \frac{T_{13}}{R'_3}, \quad \ldots\ldots\ldots\ldots \quad (24)$$

in which R'_3 is the radius of the pitch circle of spur-gear 3 and F_{34} the component of the tooth pressure that is tangential to this pitch circle.

The Pressure between the Threads of Worm 4 and the Teeth of Worm-Wheel 5.--Here, just as in the first case, we have

$$T_{15} = R_5 F_{45},$$

or

$$F_{45} = \frac{T_{15}}{R_5} = \frac{W_3 R'_5}{R_5}, \quad \ldots\ldots\ldots\ldots \quad (25)$$

wherein F_{45} is the component of the pressure between the worm and wheel that is tangential to the pitch circle of the wheel, the circle whose pitch radius is R_5.

Problem XII-10.--This problem deals with a 3-link mechanism, Fig. XII-10, containing two screw pairs and one sliding pair. Link 2, the figure shows, is paired to each of the other two links of the mechanism by a screw pair, and links 1 and 3 are paired to each other by a sliding pair. Thus link 2 has helical motion relative to links 1 and 2 and link 3 complanar motion relative to link 1.

The device is known as a differential-screw mechanism. But the student should observe that this name is not applicable unless the axial travel of link 3 relative to link 1 is proportional to the *difference* and not to the *sum* of the leads of the two screws. If the travel is proportional to the sum rather than to the difference of the two leads, the mechanism is not a *differential* but a *compound* screw mechanism. Our analysis will show that the axial travel of link 3 is proportional to the difference of the two travels and thus that the mechanism is a differential, not a compound screw mechanism.

Derivation of the Fundamental Equation.--The axial travel of link 3 relative to link 1 is obviously the resultant of the travel of link 3 relative to link 2 and the travel of link 2 relative to link 1. The fundamental equation of the solution, therefore, is the relation connecting these three axial travels. If now we let

T_{31} = Travel in inches of link 3 relative to link 1,
T_{21} = " " " " " 2 " " " 1,
T_{32} = " " " " " 3 " " " 2,

it can be shown that

$$T_{31} = T_{21} + T_{32} \quad . . . . . \quad (1)$$

holds for all possible relative arrangements of the three axial travels, provided we observe the three axiomatic conditions

$$\begin{aligned} T_{31} &= -\,T_{13}, \\ T_{21} &= -\,T_{12}, \\ T_{32} &= -\,T_{23}. \end{aligned} \quad . . . . . \quad (2)$$

To show that equation (1) holds for all possible conditions, fix the attention on three *coincident* axes, 1, 2, 3, each of indefinite length, the first attached to link 1, the second to link 2, and the third to link 3. Anywhere on axis 1 fix a point A; and coincident with point A fix to axes 2 and 3 points B and C, respectively. Then, obviously, after motion has continued for time t, segment AC will be the axial travel of link 3 relative to link 1, AB the axial travel of link 2 relative to link 1, and BC the axial travel of link 3 relative to link 2. Thus

$$\begin{aligned} T_{31} &= AC, \\ T_{21} &= AB, \\ T_{32} &= BC, \end{aligned} \quad . . . . . \quad (3)$$

clearly, moreover from (2) and (3)

$$\begin{aligned} T_{13} &= CA, \\ T_{12} &= BA, \\ T_{23} &= CB. \end{aligned} \quad . . . . . \quad (4)$$

Now consider Fig. XII-10a which shows the six possible relative arrangements of three points A, B, C on a straight line; or, what amounts to the same thing, shows the six possible relative arrangements of the three associated axial travels AC, AB, and BC.

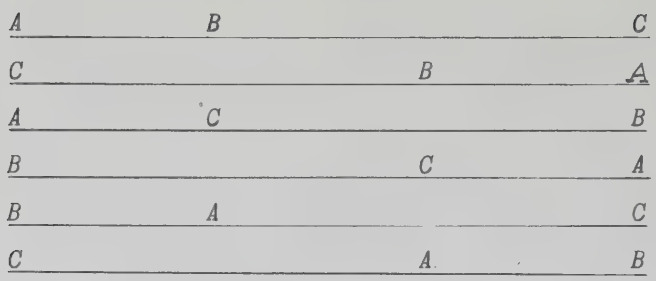


Fig. XII-10a

For the first linear arrangement of the points we have evidently

$$AC = AB + BC,$$

that is, equations (3),

$$T_{31} = T_{21} + T_{32}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (I)$$

For the second arrangement, similarly,

$$CA = CB + BA,$$

or, equations (4),

$$T_{12} = T_{23} + T_{12}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (II)$$

Similarly for the third, fourth, fifth, and sixth arrangements, respectively:

$$AB = AC + CB,$$

$$T_{21} = T_{31} + T_{23}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (III)$$

$$BA = BC + CA,$$

$$T_{12} = T_{32} + T_{13}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (IV)$$

$$BC = BA + AC,$$

$$T_{32} = T_{12} + T_{31}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (V)$$

$$CB = CA + AB,$$

$$T_{23} = T_{13} + T_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (VI)$$

That by the algebraic law of transposition and axioms (2) any one of the above six equations can be transformed into any other one it is easy to show. Let us, for example, transform V into I. Transposing T_{12} of (V) to the left-hand side of the equation, we have

$$T_{32} - T_{12} = T_{31};$$

but, axioms (2),

$$-T_{12} = T_{21},$$

therefore

$$T_{32} + T_{21} = T_{31},$$

which is equation (I). Thus we conclude that equation (1) holds for all possible relations of the six (if sense is taken into account) axial travels.

Solution of Problem XII-10.--The given data of the problem are evidently T_{31}, L_{23} and $N_{21}t$, where L_{23} is the lead in inches of the screw pair connecting links 2 and 3, N_{21} the number of turns per minute of link 2 relative to link 1, and t the time in minutes during which link 2 turns relative to link 1 at the rate of N_{21} turns per minute. The algebra of our problem is, therefore, to eliminate from equation (1) the unknown quantities T_{21} and T_{32} and put in their stead the given data. Now, manifestly,

$$T_{21} = (N_{21}t)L_{21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

and

$$T_{32} = (N_{32}t)L_{23}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

in which L_{21} and L_{23} are the leads in inches of the two screw pairs, and $(N_{21}t)$ and $(N_{32}t)$, respectively, the number of turns in t minutes of link 2 relative to link 1 and $(N_{32}t)$ the number of turns in the same time of link 3 relative to link 2. But, inspection of the figure shows that

$$N_{21} = N_{23} = -N_{32}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

therefore equation (6) becomes

$$T_{32} = -(N_{21}t)L_{23}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Now substituting from equations (5) and (8) in equation (1) we have

$$T_{31} = (N_{21}t)L_{21} - (N_{21}t)L_{23},$$

$$= (L_{21} - L_{23})N_{21}t, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

or, solving for the required lead,

$$L_{21} = \frac{T_{31}}{N_{21}t} + L_{23}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

a formula with which it is easy to solve the given problem.

Without, however, some rule by which the signs of the terms of equation (10) can be fixed for any given case it would be of little service to us.

Rule of Signs for Equation (10).--The rule consists of three parts, as follows:

Part I.--(a) If the axis of the screw is vertical, the observer is supposed to stand above the mechanism and look down on it; and then clockwise rotation will be considered positive and counterclockwise rotation therefore negative.

(b) If the axis of the screw is horizontal, the observer is supposed to stand on the left of the mechanism and look at it; and then, again, clockwise rotation will be considered positive and counterclockwise rotation negative.

Part II.--The lead of a right-hand thread will be considered positive; the lead of a left-hand one negative.

Part III.--If we adopt the two preceding conventions, then, evidently *downward* travel and travel from *left to right* must be considered positive, while *upward* travel and travel from *right to left* must be considered negative.

The student should observe that the first two parts of the rule of signs are independent of each other and can be arbitrarily fixed as we please, but that the third part of the rule depends on the first two parts and cannot, therefore, be arbitrarily fixed.

Let us now use formula (10) to obtain the numerical answers to the problem. It is merely a matter of substituting in formula (10) the given data of the problem. In using this equation we should be careful to observe that there will be two values for L_{21}; that is, two answers to the problem, because link 2 may be turned in either a clockwise or a counterclockwise sense relative to link 1, the link of reference.

First Solution, Link 2 Turning Clockwise.--In this case we must substitute -5.5" for T_{31}, +20 for $N_{21}t$, and +1/2" for L_{23}, because (1) the travel of 3 is 5.5" *upward*, (2) the rotation of 2 is 20 turns *clockwise*, and (3) the screw-pair connecting links 2 and 3 is *right-hand* and the lead of it 1/2". Making these substitutions and solving for L_{21}, we get

$$L_{21} = +\frac{9}{40};$$

that is, to meet the conditions of this case the lead of the screw pair connecting links 1 and 2 must be 9/40" and the thread must be right-hand.

Second Solution, Link 2 Turning Counterclockwise.--For this case substitutions (1) and (3) remain the same as for the other case, but substitution (2) is now -20, because link 2 turns counterclockwise relative to link 1, the link of reference. This change in the data changes the value of L_{21} from +9/40" to +31/40".

The student should be sure to notice that in stating the problem the authors have given us some superfluous information. They should not have given us the "hand" of the screw pair connecting links 1 and 2 because, as our solutions show, that is implicit (see the definition of this word in Webster's New International Dictionary) in the other given data.

Writing equation (9) in the form

$$T_{31} = (N_{21}t)(L_{21} - L_{23}) \quad \ldots\ldots\ldots\ldots\ldots \quad (9')$$

we see: (1) that if L_{21} and L_{23} are either both positive or both negative, that is, if both screw pairs are either both right-hand or both left-hand, T_{31} will depend upon the *difference* of the two leads; (2) that if the sign of L_{21} is opposite to the sign of L_{23}, that is, if one screw pair is right-hand and the other left-hand, T_{31} will depend upon the *sum* of the two leads. In the first case, therefore, the mechanism is a *differential screw* mechanism; in the second, a *compound screw* mechanism.

Problem XII-13.--This also is a differential screw mechanism containing two screw pairs, but here the link (link 4) that corresponds to link 3 of the mechanism of Problem XII-10 is not paired to link 1 by a sliding pair but by an*imaginary* screw pair; that is, links 1 and 4 have constrained helical relative motion but are not actually paired to each other by a real screw pair. (We shall return to this point later in the explanation.) Part of the differential effect to which screw 4 of this unibasal quartic gear train, 1, 2, 3, 4, is subjected is due to screw pairs 1, 2 and 2, 4, and part is due to gear train 1, 2, 3, 5, which forms a part of the mechanism.

The mechanism consists, as the figure shows, of five links paired to each other as follows: Link 1 and 2 by a screw pair, links 1 and 3 by a turning pair, links 1 and 5 by a turning pair, links 4 and 5 by a sliding pair, links 2 and 4 by a screw pair, links 2 and 3 by a higher pair, and links 3 and 5 by a higher pair.

The problem here is to calculate how much and in which sense--*up* or *down*--screw 4 moves when the graduated dial is turned counterclockwise (observe the arrow attached to the dial) 1/100th of a turn.

Solution of Problem XII-13.--For the relative axial travels of links 1, 2, and 4, neglecting the rotary relative motion of these pieces, we have, just as in Problem XII-10 the equation

$$T_{41} = T_{21} + T_{42}. \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Thus the algebra of our problem is the elimination of T_{21} and T_{42} from equation (1) with the aid of the known data. Now the known data are (1) the four Z's, that is, the numbers of teeth on the four gears, (2) the two leads L_{21} and L_{24}, and (3) $N_{31}t$, the amount of angular motion of link 3 relative to link 1. Proceeding with this elimination, we have, first

$$T_{21} = (N_{21}t)L_{21}, \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

and from the cubic spur-gear mechanism 1, 2, 3,

$$\frac{N_{21}}{N_{31}} = \frac{N_{21}t}{N_{31}t} = -\frac{Z_3}{Z_2}; \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

thus, combining equations (2) and (3),

$$T_{21} = -\frac{Z_3}{Z_2}(N_{31}t)L, \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

an equation that expresses axial travel T_{21} in terms of known quantities.

To express T_{42} of equation (1) in terms of known things is a little more difficult, but not much more so. First, evidently, as in equation (2),

$$T_{42} = (N_{42}t)L_{24}; \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

and by Theorem IX, taking into account only the angular motion of links 1, 2, 4, we have

$$N_{42} = N_{12} + N_{41};$$

or, multiplying through by t,

$$\begin{aligned} N_{42}t &= N_{12}t + N_{41}t \\ &= N_{41}t - N_{21}t. \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

Also from the cubic spur gear mechanism 1, 3, 5, noticing that since screw 4 and gear 5 are splined to each other, the *angular* motions of these two pieces are equal and thus that $N_{51} = N_{41}$,

$$\frac{N_{51}t}{N_{31}t} = \frac{N_{41}t}{N_{31}t} = -\frac{Z'_3}{Z_5}. \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

Then from equation (6), with the aid of equations (3) and (7), eliminating $N_{21}t$ and $N_{41}t$ in terms of known things, we get

$$\begin{aligned} N_{42}t &= -\frac{Z'_3}{Z_5}(N_{31}t) + \frac{Z_3}{Z_2}(N_{31}t) \\ &= \frac{Z_3Z_5 - Z_2Z'_3}{Z_2Z_5}(N_{31}t). \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

Now substituting this value of $N_{42}t$ in equation (5) there results the equation

$$T_{42} = \frac{Z_3Z_5 - Z_2Z'_3}{Z_2Z_5}(N_{31}t)L_{24}, \quad \ldots\ldots\ldots\ldots \quad (9)$$

which expresses axial travel T_{42} in terms of given quantities.

Finally, eliminating T_{21} and T_{42} from equation (1) with the help of equations (4) and (9), we get

$$T_{41} = \left\{\frac{Z_3Z_5 - Z_2Z'_3}{Z_2Z_5}L_{24} - \frac{Z_3}{Z_2}L_{21}\right\}N_{31}t, \quad \ldots\ldots\ldots \quad (10)$$

a formula that expresses the required axial travel of screw 4 relative to link 1 in terms of the eight given data of the problem.

To get the numerical value of T_{41}, that is, the answer to the problem, it is now merely a matter of substituting the eight given numerical values in equation (10), carefully observing as we do so the rule of signs stated in connection with Problem XII-10. Thus, for Z_2, Z_3, Z'_3, Z_5, L_{21}, L_{14}, and $N_{31}t$, the given quantities, substituting 200, 18, 12, 120, -1/42, 1/40, and -1/100, respectively, we obtain

$$T_{41} = -\left\{\frac{18\times120 - 200\times12}{200\times120} \times \frac{1}{40} + \frac{18}{200} \times \frac{1}{42}\right\}\frac{1}{100}$$

$$= -\left\{-\frac{1}{4000} + \frac{3}{1400}\right\}\frac{1}{100}$$

$$= -\left\{-\frac{7}{28000} + \frac{60}{28000}\right\}\frac{1}{100}$$

$$= -\frac{58}{2,800,000}\text{"};$$

that is, the travel of screw 4 relative to link 1 for 1/100th of a turn of link 3 counterclockwise relative to link 1 is 53/1,800,000" *upward.*

Discussion of Equation (10).--1. We have said that links 1 and 4 are paired to each other by an imaginary screw pair, and this is obviously the case, because these two pieces have helical relative motion; that is, each inch of axial motion of screw 4 relative to link 1 is accompanied by a certain constant amount of rotary motion. It is easy to find the expression for the lead of the imaginary screw pair connecting these two pieces. If from equation (10) we eliminate $N_{31}t$ with equation (7) we get

$$T_{41} = \left\{\frac{Z_3Z_5 - Z_2Z'_3}{Z_2Z_5}L_{24} - \frac{Z_3}{Z_2}L_{21}\right\}\left(-\frac{Z_5}{Z'_3}N_{41}t\right)$$

$$= -\left\{\frac{Z_3Z_5 - Z_2Z'_3}{Z_2Z'_3}L_{24} - \frac{Z_3Z_5}{Z_2Z'_3}L_{21}\right\}(N_{41}t), \quad \ldots\ldots\ldots \quad (11)$$

from which we see that the factor in the parenthesis is the lead of this imaginary screw pair, because this factor multiplied by the number of turns of screw 4 relative to link 1 is equal to the axial travel of 4 relative to 1. The numerical value of this lead is therefore given by the equation

$$L_{41} = -\left\{\frac{Z_3Z_5 - Z_2Z'_3}{Z_2Z'_3}L_{24} - \frac{Z_3Z_5}{Z_2Z'_3}L_{21}\right\}. \quad \ldots\ldots\ldots \quad (12)$$

$$L_{41} = -\frac{18\times120 - 200\ 12}{200\times12} \times \frac{1}{40} - \frac{18\times120}{200\times12} \times \frac{1}{42}$$

$$= -\frac{53''}{2800}.$$

The sign coming out negative means that the imaginary screw pair is left-hand.

The result that we have just derived shows that if link 2 were eliminated from the mechanism and links 1 and 4 paired to each other directly by a screw pair whose lead is 53/2800", left-hand, screw 4 would move up 53/2,800,000" for 1/100th of a turn counterclockwise of the graduated dial relative to link 1. Let us check this statement with the actual figures. Equation (7) shows that 1/100th of a turn of the dial counterclockwise causes 1/1000 of a turn of screw 4 clockwise; therefore the product of 1/1000 and -53/2800 should be equal to -53/2,800,000; and obviously it is.

This arithmetical illustration suffices to show very strikingly how it is possible to realize the mechanical equivalent of a screw with a lead that is almost microscopically small--in this case only 53/2800". (The student should be sure to study this illustration carefully.)

2. A little further study of equation (11) shows that we can design this mechanism so that it is essentially a differential-screw mechanism; in other words, we can design it so that it is kinematically exactly like the mechanism of Problem XII-10. If, first, we so select the four Z's of the gear train that the coefficient of L_{24} is positive, and also make the two screw pairs either both right-hand or both left-hand, one of the terms in the parenthesis of equation (11) will be positive and the other negative, and the form of the resulting equation will be exactly like that of equation (9), Problem XII-10. If, secondly, we select the Z's so that the coefficient of L_{24} is negative, and choose the two leads so that L_{21} is positive and L_{24} negative or vice versa, again one of the terms of the parenthesis will be positive and the other negative and the form of the resulting equation will be the same as that of equation (9), Problem XII-10.

3. These considerations serve to show that the effect of the gear-train part of the mechanism of this Problem XII-13 is merely to modify--increase or decrease--the values of the two leads, L_{21} and L_{24}. We may, in other words, think of the mechanism of this problem as consisting of only three pieces, 1, 2, and 4; links 1 and 2 being paired to each other by a screw pair whose lead is--$Z_3Z_5L_{21}/Z_2Z'_3$ and links 2 and 4 being paired to each other by a screw pair whose lead is--$(Z_3Z_5 - Z_2Z'_3)L_{24}/Z_2Z'_3$.

Problem XII-12.--A comparison of Figs. XII-12 and XII-13 will show that the only difference between the mechanisms which they represent is that in the first links 1 and 2 are paired to each other by a turning-pair whereas in the second these two links are paired by a screw pair. Hence formula (10) of the Solution of Problem XII-13 becomes suitable for the solution of any kinematic problem concerning the mechanism of Fig. XII-12 by making L_{21} in this formula zero. Thus for the mechanism of Fig. XII-12

$$T_{41} = \frac{Z_3Z_5 - Z_2Z'_3}{Z_2Z_5} L_{24}(N_{31}t).$$

Problem XII-14.--A comparison of Figs. XII-13 and XII-14 shows that the mechanisms which they represent though quite unlike constructively are exactly alike kinematically; hence formula (10) of the Solution of Problem XII-13 will serve for the solution of any kinematic problem which can arise in connection with the mechanism of Fig. XII-14, provided the two figures are symbolized to correspond.

IV. Pitch Surfaces of Spur and Bevel Gears

A. Articles 8-1, 8-2, 8-3, 8-4

Rolling Cylinders of Revolution: Pitch Surfaces of Spur Gears.--In Fig. 8-1 two circles, C_1 and C_2, whose centers are 13 and 12, are shown tangent to each other at point 23. The circles represent cylinders of revolution, whose axes 13 and 12 pass through points 13 and 12 perpendicular to the plane of motion. From the geometry of the figure it is evident that the cylinders touch each other along the straight line that passes through point 23 parallel to axes 13 and 12. Call this line of tangency L_{23}, and consider any two coincident points P_2 and P_3 on it, P_2 fastened to cylinder 2 and P_3 to cylinder 3. Now if the cylinders *roll* on each other--and that is the hypothesis--these two points have no relative velocity. Thus they must have the same velocity relative to any third body--link 1, for instance. And since this is the case for all pairs of coincident points on line L_{23}, this line must, in accordance with Theorem VI, be the velocity-axis of the two cylinders; that is, the axis about which at and for the instant under consideration these two cylinders of revolution are rotating relative to each other. We designate this velocity-axis with the symbol 23.

We have said that the two cylinders *roll* on each other. In practice, when the cylinders are the pitch-surfaces of a pair of spur-gears, rolling action between the cylinders is constrained by means of teeth, which are so shaped and related that by sliding on each other they compel the cylinders to move in rolling contact.

Now if there is no slip between the cylindrical surfaces the three pieces constitute a 3-link kinematic chain, and if then we fix any one of the three links a mechanism results.

Velocity Axes.--Thus we have three rigid bodies undergoing instantaneous complanar motion, for which in accordance with Theorem VII, there are the three parallel and complanar velocity-axes 12, 13, and 23. Of these three velocity axes the first two are permanent and the third instantaneous.

Angular-Velocity Ratios. Case I: Cylinders in External Contact.--Since there are three bodies or links there are three angular-velocity ratios. They are

$$\frac{N_{21}}{N_{31}}, \quad \frac{N_{12}}{N_{32}}, \quad \frac{N_{13}}{N_{23}},$$

each of which, as we know from Theorem VIII, can be expressed in terms of the radii of the cylinders, R_2 and R_3. Thus

$$\frac{N_{21}}{N_{31}} = -\frac{23\text{-}31}{23\text{-}21} = -\frac{R_3}{R_2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{N_{12}}{N_{32}} = \frac{13\text{-}32}{13\text{-}12} = \frac{R_3}{R_2 + R_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\frac{N_{13}}{N_{23}} = \frac{12\text{-}23}{12\text{-}13} = \frac{R_2}{R_2 + R_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

If the cylinders are the pitch surfaces of spur-gears, for which Z_2 and Z_3 are the numbers of teeth and P'_{23} the circular pitch, then multiplying numerator and denominator of the left-hand side of each of the foregoing equations by $2\pi/P'_{23}$, we get

$$\frac{N_{21}}{N_{31}} = -\frac{Z_3}{Z_2} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (4)$$

$$\frac{N_{12}}{N_{32}} = \frac{Z_2}{Z_2 + Z_3} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (5)$$

$$\frac{N_{13}}{N_{23}} = \frac{Z_3}{Z_2 + Z_3} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (6)$$

Angular-Velocity Ratios. Case II: Cylinders in Internal Contact.--For this case, Fig. 8-2, making the same assumption and reasoning in exactly the same way as for Case I, we have

$$\frac{N_{21}}{N_{31}} = \frac{23\text{-}31}{23\text{-}21} = \frac{R_3}{R_2} = \frac{Z_3}{Z_2} \quad \ldots \ldots \ldots \ldots \ldots \quad (7)$$

$$\frac{N_{12}}{N_{32}} = \frac{13\text{-}32}{13\text{-}12} = \frac{R_3}{R_3 - R_2} = \frac{Z_3}{Z_3 - Z_2} \quad \ldots \ldots \ldots \ldots \quad (8)$$

$$\frac{N_{13}}{N_{23}} = -\frac{12\text{-}23}{12\text{-}13} = -\frac{R_2}{R_3 - R_2} = -\frac{Z_2}{Z_3 - Z_2}$$

$$= \frac{Z_2}{Z_2 - Z_3} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (9)$$

Problems of Rolling Cylinders of Revolution.--Since cylinders of revolution are extensively used as the pitch-surfaces of spur-gears and friction drives, there are many important problems of kinematic design concerning them. We shall now show how to solve some of these problems.

Problem I.--Two cylinders of revolution are the pitch surfaces of two spur-gears. The center-distance for the gears is d inches and the smaller of the gears makes m turns clockwise while the larger makes 1 turn counterclockwise; determine R_2 and R_3, the radii of the two cylinders, R_2 being the smaller of the two.

1. Analytical Solution.--(1) Since the two gears turn in opposite senses they mesh externally and therefore, equation (1).

$$\frac{N_{21}}{N_{31}} = -\frac{R_3}{R_2} = -\frac{m}{1} \quad \ldots \ldots \ldots \ldots \ldots \quad (10)$$

(2) Since, moreover, the gears are in external mesh, the center-distance is equal to the sum of the radii, that is,

$$R_2 + R_3 = d \quad \ldots \ldots \ldots \ldots \ldots \quad (11)$$

Therefore, solving equations (10) and (11) for R_2 and R_3,

$$R_2 = \frac{d}{m + 1} \quad \ldots \ldots \ldots \ldots \ldots \quad (12)$$

$$R_3 = \frac{md}{m + 1} \quad \ldots \ldots \ldots \ldots \ldots \quad (13)$$

2. Graphical Solution.--In Fig. 8-1 make center-distance 12-13 equal to d inches, to scale if necessary. Draw *12-Q* and *13-P* perpendicular to and on *opposite* sides of the center-line. Lay off *13-P* equal to m units and *12-Q* equal to 1 unit. Join P and Q, and note the point, *23*, in which PQ cuts *12-13*. Then, obviously, triangles *(13)P(23)* and *(12)Q(23)* are similar, and thus

$$\frac{23\text{-}13}{23\text{-}12} = \frac{R_3}{R_2} = \frac{13\text{-}P}{12\text{-}Q} = \frac{m}{1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

that is,

$$\frac{R_3}{R_2} = \frac{m}{1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

which shows that this construction so locates point *23* as to satisfy equation (10).

That the construction satisfies equation (11) is obvious.

Comments: (1) Of course it is not necessary for *13-P* and *12-Q* to be perpendicular to center-line *12-13*. It is sufficient for them to be only parallel. It is convenient, however, when working on a drawing-board to make them perpendicular to *12-13*.

(2) The student should understand that while this graphical construction is a helpful check on algebraic results it is not accurate enough in most cases to give satisfactory design dimensions.

(3) In making this construction the student should be careful to lay off the segment containing the larger number of units from the center of the slower-moving gear, because the proportion $N_{21}/N_{31} = -m/1$ is inverse.

Problem 2.--Same conditions as for Problem 1 except that the gears turn in the same sense, not in opposite senses, relative to baselink 1.

1. Analytical Solution.--(1) Since the gears turn in the same sense the smaller must mesh the larger internally. For this case, Fig. 8-2, therefore, we have, equation (4).

$$\frac{R_3}{R_2} = \frac{m}{1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

(2) Since the gears are in internal mesh we have, moreover,

$$R_3 - R_2 = d \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

Thus, equations (16) and (17),

$$R_2 = \frac{d}{m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

and

$$R_3 = \frac{md}{m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

2. Graphical Solution.--The procedure, the explanation, and the comments are exactly the same here as in the graphical solution of Problem 1 except that segments *13-P* and *12-Q* must for this case be laid off on the *same* side of the center-line.

Caution: In the graphical solutions of these two problems, the scale to which *13-P* and *12-Q* are laid off should be so chosen as to make the angle between lines PQ and *12-13*

at least as great as 30°; otherwise their point of intersection, velocity-center *23*, will be inaccurately determined.

The variable-speed friction drive for parallel shafts, Fig. 8-2, is not, strictly speaking, a case of rolling cylinders of revolution; nevertheless it is convenient to solve the kinematic problem concerning it in this place.

The student will do well to pay careful attention to the next two problems because their solutions afford excellent examples of the usefulness and power of mathematical reasoning in the kinematic analysis of mechanisms.

Problem 3.--(a) For the configuration of the mechanism shown find in terms of D and R_3 the sign and magnitude of angular-velocity ratio N_{21}/N_{31}.

(b) If $N_{21}/N_{31} = -n$; find the position of friction disk 4.
(c) Locate velocity-axis *23* when disk *4* is R_3 inches below velocity-axis *13*.

Analytical Solution. Part (a).--(1) With friction disk *4* anywhere between velocity-axes *12* and *13* disks *2* and *3* obviously turn in the *same* sense relative to baselink 1. Thus for such a position of disk *4* N_{21}/N_{31} is positive. With friction disk *4* anywhere above velocity-axis *12* or anywhere below velocity axis *13* these two disks turn in *opposite* senses relative to baselink 1. Hence for these two positions of disk *4* N_{21}/N_{31} is negative. The sign of N_{21}/N_{31} for this part of the problem is therefore *positive*.

(2) Now if there is no slip between disk *4* and either of the other disks

$$Q_2V_{21} = Q_4V_{41} = P_4V_{41} = P_3V_{31}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

that is,

$$Q_2V_{21} = P_3V_{31}. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

and

$$2\pi N_{21}R_2 = 2\pi N_{31}R_3 \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

A result which, cancelling the common factor, we may write in the form of a proportion

$$\frac{N_{21}}{N_{31}} = \frac{R_3}{R_2}. \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

the sign of the right-hand side of which is positive, since, as has been pointed out, disks *2* and *3* turn in the same sense relative to baselink 1 when disk *4* is anywhere between, as it is in the figure, velocity axes *12* and *13*.

(3) From the figure, furthermore,

$$R_2 = D - R_3; \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

therefore, equations (4) and (5),

$$\frac{N_{21}}{N_{31}} = \frac{R_3}{D - R_3}, \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

which, as required, gives N_{21}/N_{31} in terms of R_3 and D.

Part b.--(1) Since for this case N_{21}/N_{31} is negative, disk *4* must be located either *above* velocity-axis *12* or *below* velocity-axis *13*.

Reasoning for this case just as we did for the case in which disk *4* was between velocity-axes *12* and *13*, we find

$$\frac{N_{21}}{N_{31}} = -\frac{R_3}{R_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

the negative sign of which is necessary because disks *2* and *3* turn in opposite senses relative to baselink 1. But by the condition of the problem

$$\frac{N_{21}}{N_{31}} = -n; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

therefore, equations (7) and (8),

$$\frac{R_3}{R_2} = \frac{n}{1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

a result which shows that R_3 will be greater or less than R_2 according as n is greater or less than *1*. But when, as is the case for this part of the problem, disk *4* is not between velocity-axes *12* and *13* the only way, evidently, in which R_3 can be (1) numerically greater than or (2) numerically less than R_2 is for disk *4* to be *above* velocity-axis *12* in the former case and *below* velocity-axis *13* in the latter case. Hence disk *4* must lie *above* velocity-axis *12* if n is greater than *1*, and *below* velocity-axis *13* if n is less than *1*.

(2) From the figure it is evident that when disk *4* is *above* velocity-axis *12*, that is, when n is greater than *1*,

$$R_3 - R_2 = D, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

and when disk *4* is *below* velocity-axis *13*, that is, when n is less than *1*.

$$R_2 - R_3 = D. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

For the former case, in which n is greater than *1*, we have therefore, equations (9) and (10),

$$R_2 = \frac{D}{n-1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

$$R_3 = \frac{nD}{n-1}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

and for the latter case, in which n is less than *1*, equations (9) and (11),

$$R_2 = \frac{D}{1-n}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

$$R_3 = \frac{nD}{1-n}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

Part (c).--(1) To locate velocity-axis *23* it is necessary to remember, (1) that it is parallel to and in the same plane with velocity-axes *12* and *13* and lies or does not lie between them according as N_{21}/N_{31} is negative or positive, (2) that any point on it has the same velocity relative to baselink *1* whether we consider the point to belong to disk *2* or to disk *3*.

Let S_2S_3 designate two such coincident points anywhere on velocity-axis *23*, and let R'_2 and R'_3 designate the distances of these points from velocity-axes *12* and *13*, respectively.

Then

$$S_2V_{21} = S_3V_{31}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

and

$$2\pi R_2' N_{21} = 2\pi R_3' N_{31},$$

that is,

$$\frac{N_{21}}{N_{31}} = \frac{R_3'}{R_2'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

the sign of which should, of course, be negative despite the fact that the process by which we have derived the equation does not automatically yield the sign. It does not automatically yield the sign, it should be noted, because the reasoning is somewhat less complicated if we do not give R_2' and R_3' positive and negative senses. Thus we may write

$$\frac{N_{21}}{N_{31}} = -\frac{R_3'}{R_2'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

Now for this case we have another expression for N_{21}/N_{31}, equation (7). Hence equations (7) and (18)

$$\frac{R_3}{R_2} = \frac{R_3'}{R_2'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

Moreover, since N_{21}/N_{31} is negative velocity axis *23*, as we have already pointed out, lies between velocity-axes *12* and *13*; thus

$$R_2' + R_3' = D. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

Now solving these three simultaneous equations, (19), (20), (11) for R_3', we get

$$R_3' = \frac{DR_3}{2R_3 + D}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

which gives the distance of velocity-axis *23* above velocity-axis *13* in terms of the known things, R_3 and D.

The student should be sure to derive corresponding formulas for the other two cases, namely: (1) when disk *4* lies *above* velocity-axis *12*, and (2) when disk *4* lies *between* velocity-axes *12* and *13*.

Graphical Solutions. A. To locate the position of disk *4* when $N_{21}/N_{31}(= \pm n)$ and D are given, proceed as follows:

1. $N_{21}/N_{31} = +n$. Lay off on *opposite* sides of velocity-axis *14*, from points S and N, along velocity-axes *13* and *12*, respectively, segments ST_1 *(= n units)* and NM *(= 1 unit)*. The point Q_1 in which MT_1 cuts velocity-axis *14* gives the location of disk *4*.

2. $N_{21}/N_{31} = -n$. Same procedure except segments ST and NM are laid off on the same side of velocity-axis *14*. Point O, the intersection of *14* and TM gives the location of disk 4.

B. To locate velocity-axis *23* when N_{21}/N_{31} $(= \pm n)$ and D are given, proceed as follows:

1. $N_{21}/N_{31} = +n$. For this case draw velocity-axis *23* through point O (located as explained in A2) parallel to velocity axes *12* and *13*.

2. $N_{21}/N_{31} = -n$. For this case draw velocity-axis *23* through point Q_1 (located as explained in A1) parallel to velocity-axes *12* and *13*.

The proof of A2 and B1 follows at once from the fact that triangles ONM and OST are similar; the proof of A1 and B2 follows from the fact that triangles O_1NM and O_1ST_1 are similar.

Problem 4.--This problem concerns the variable-speed friction drive for intersecting axes shown in Fig. 8-4. In the figure the angle between the shafts is 90° but this angle may have any value between 0° and 90°. Friction disk 4, which contacts the two conoids, *2* and *3*, at points P_2 and Q_3,, respectively, can be turned about vertical axis *15* into any position from horizontal to vertical and then, by means not shown in the figure, locked.

If N_{21} is constant N_{31} is obviously a maximum when disk 4 is horizontal, that is, when angle θ is zero. On the other hand, when disk 4 is vertical, that is, when θ is 90°, N_{31} will, it is evident, be a minimum. For each value of θ between 0° and 90° N_{31} will take some value between its upper and lower limit.

Mathematically, our problem is to express angular-velocity ratio N_{21}/N_{31} in terms of the angle θ and the constants of the machine, R'_2 and R'_4, all of which are clearly indicated in the figure.

(1) If there is no slip between disk 4 and either of the conoids, then

$$P_2V_{21} = P_4V_{41} = Q_4V_{41} = Q_3V_{31}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

that is,

$$P_2V_{21} = Q_3V_{31}, \quad \ldots\ldots\ldots\ldots \quad (2)$$

and therefore

$$2\pi R_2N_{21} = 2\pi R_3N_{31}, \quad \ldots\ldots\ldots\ldots \quad (3)$$

which may be written

$$\frac{N_{21}}{N_{31}} = \frac{R_3}{R_2}. \quad \ldots\ldots\ldots\ldots \quad (4)$$

But from the figure, obviously, since triangles *(15)*P_2T and *(15)*Q_3S are similar and equal,

$$R_2 = R'_2 - R_4 \sin\theta. \quad \ldots\ldots\ldots\ldots \quad (5)$$

and

$$R_3 = R'_3 - R_4 \cos\theta. \quad \ldots\ldots\ldots\ldots \quad (6)$$

Therefore, equation (4), (5), (6),

$$\frac{N_{21}}{N_{31}} = \frac{R'_3 - R_4\cos\theta}{R'_2 - R_4\sin\theta}, \quad \ldots\ldots\ldots\ldots \quad (7)$$

which expresses angular velocity ratio N_{21}/N_{31} in terms of angle θ and the three constants of the machine.

Comments: 1. As shown in the figure the two conoids are exactly alike. This, however, though economically desirable, is not a necessary design condition.

2. There is a friction drive on the market--trade name "TRANSITOR Q"--which somewhat resembles this device, despite the fact that the driving shaft is in line with the driven shaft. The makers claim that slippage is completely eliminated in it.

3. Such mechanisms as the two with which we have just dealt though much used and very useful in some applications are not fit for applications where slippage is inadmissable.

4. This is really a screw mechanism because for any given position of disk 4, except the 45-degree position, conoids *2* and *3* have special screw motion relative to each other.

Each of them, however, has complanar motion relative to frame 1, since each is paired to the frame by a turning pair. When axes *12, 13, 14* meet in a point the two conoids have spheric relative motion.

In this connection the student would find it worth while--as a kinematic exercise, purely--to try to solve this problem: Given $\theta(= 45^{\circ})$ and $N_{21}(= 600)$ to find N_{23}. Prove that the answer is $N_{23} = 600\sqrt{2}$.

5. Let the student consider how the pressure between *2* and *3* at P_2P_4 and the pressure between *3* and *4* at Q_3Q_4 could be automatically increased as the resistance torque acting on shaft *3* is increased. The purpose in doing this would be to prevent slippage at P_2P_4 and Q_3Q_4.

B. Article 8-5, 8-6, 8-7, 8-8

Rolling Cones of Revolution: Pitch Surfaces of Bevel Gears.--In Fig. 8-5 portions or frustra of two cones of revolution are represented. They are tangent to each other along the line 23 and their apices coincide. They are paired to baselink 1 by turning pairs 12 and 13. Thus links 2 and 3 have complanar motion relative to link 1 and--if the two cones *roll* on each other--spheric motion relative to each other.

When the cones are the pitch surfaces of bevel gears they are provided with teeth to prevent slip at the line of contact between them. The teeth are so shaped and related that by sliding on each other they compel the two cones to roll on each other.

Velocity Axes.--The three pieces or links--the two gears and the baselink--constitute a 3-link spheric mechanism of the greatest usefulness to industry. Since there are three links in the mechanism there are three velocity-axes, one velocity-axis for each pair of links. The velocity axes are 12, 13, and 23, the first two of which are *permanent*, and the third *instantaneous*. It is perfectly obvious that 12 and 13 are velocity-axes, but it not quite so obvious that 23 is. Yet it is not difficult to see that line 23 is the velocity-axis of links 2 and 3. For if the two tangent cones *roll* on each other then any point P_2 of link 2, on the line of contact between the cones, has the same linear velocity relative to link 1 as the coincident point P_3 of link 3. But if these two points have the same velocity relative to link 1, they have no velocity relative to each other. Thus this line of contact has no velocity at the instant relative to either of the cones. That is to say, the line of contact is velocity-axis 23, the axis, in other words, about which at the instant links 2 and 3 are turning relative to each other.

It is obvious that the three velocity-axes 12, 13, and 23 not only lie in the same plane but also meet in point O in the figure.

Angular Velocity Ratios.--Since there are three links in the mechanism there are three cubic angular velocity ratios associated with it. They are:

$$\frac{N_{21}}{N_{31}}, \quad \frac{N_{12}}{N_{32}}, \quad \frac{N_{13}}{N_{23}}$$

To make a complete kinematic analysis of this mechanism it is necessary to find expressions for each of these ratios. This we now proceed to do.

(a) *The Expression for Angular Velocity Ratio* N_{21}/N_{31}.--The Symbolism of this ratio shows that baselink 1 is the link of reference for it, and links 2 and 3 the moving links.

Let P_2P_3 represent *any* two coincident points on velocity-axis 23, P_2 belonging to link 2, P_3 to link 3. The linear velocity of point P_2 relative to link 1 is clearly

$$P_2V_{21} = 2\pi R_2 N_{21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

and the linear velocity of point P_3, relative to link 1 is

$$P_3V_{31} = 2\pi R_3 N_{31}, \quad \ldots \ldots \ldots \ldots \ldots \quad (2)$$

but since the two cones roll on each other,

$$P_2V_{21} = P_3V_{31}, \quad \ldots \ldots \ldots \ldots \ldots \quad (3)$$

that is,

$$2\pi R_2 N_{21} = 2\pi N_{31} R_3, \quad \ldots \ldots \ldots \ldots \ldots \quad (4)$$

or

$$\frac{N_{21}}{N_{31}} = \frac{R_3}{R_2}, \quad \ldots \ldots \ldots \ldots \ldots \quad (5)$$

in which R_2 and R_3 are the perpendiculars from P_2P_3 to velocity-axes 12 and 13, respectively, that is, radii of the pitch cones at their large ends.

Now let P'_{23} = Circular pitch of the two gears at their large ends,
Z_2 = Number of teeth on gear 2,
Z_3 = Number of teeth on gear 3.

Then multiplying numerator and denominator of the right-hand side of equation (5) by $2\pi/P'_{23}$, we have

$$\frac{N_{21}}{N_{31}} = \frac{R_3}{R_2} = \frac{\frac{2\pi R_3}{P'_{23}}}{\frac{2\pi R_2}{P'_{23}}} = \frac{Z_3}{Z_2}. \quad \ldots \ldots \ldots \ldots \quad (6)$$

That is to say, the number of turns per minute of gear 2 relative to baselink 1 is to the number of turns per minute of gear 3 relative to baselink 1 as the number of teeth on gear 3 is to the number of teeth on gear 2.

The student should be sure to observe: (1) That proportion (6) is inverse, that is, Z_3 is the number of teeth on the gear whose angular velocity is N_{31}, and Z_2 the number of teeth on the gear whose angular velocity is N_{21}. (2) That equation (6) for bevel gears is of exactly the same mathematical form as the corresponding equation for spur-gears.

Equation (6) gives us two expressions for the angular velocity ratio N_{21}/N_{31}. There is a third expression for this ratio which is of use in analyzing this important 3-link mechanism. This third expression will now be derived.

In Fig. 8-5 locate point P *anywhere* on velocity-axis 23 and through it draw a line parallel to velocity-axis 12, cutting velocity-axis 13 in point Q. Then

Angle $POQ = \alpha_3$, the half pitch-cone angle of gear 3,
Angle $OPQ = \alpha_2$, the half pitch-cone angle of gear 2,
Angle $PQO = 180 - \alpha_1$, the supplement of the acute angle between velocity-axes 12 and 13.

Now dividing R_3 and R_2, equation (6), by OP_2, the common hypotenuse of right-angle triangles OS_3P_3 and OQ_2P_2, we get

$$\frac{N_{21}}{N_{31}} = \frac{R_3}{R_2} = \frac{\frac{R_3}{OP_2}}{\frac{R_2}{OP_2}} = \frac{\sin \alpha_3}{\sin \alpha_2}. \quad \ldots \ldots \ldots \ldots \quad (7)$$

Moreover, applying the law of sines to triangle OPQ,

$$\frac{PQ}{QO} = \frac{\sin \alpha_3}{\sin \alpha_2}. \quad \ldots \ldots \ldots \ldots \ldots \quad (8)$$

Therefore, equations (7) and (8),

$$\frac{N_{21}}{N_{31}} = \frac{PQ}{QO}, \quad \ldots \ldots \ldots \ldots \ldots \quad (9)$$

which tells us that to whatever scale of angular velocity PQ represents angular velocity N_{21}, QO to the same scale represents angular velocity N_{31}. In other words, equation (9) means that if we lay off along velocity-axis 13 vector OQ to scale to represent angular velocity N_{31} and through Q, thus determined draw QP to the same scale will represent angular velocity N_{21}.

Equation (9), it should be noticed, leads directly to the following construction for locating velocity axis 23 and thus determining angles α_2 and α_3, when α_1, the angle between the axes of gears 2 and 3, and N_{21}/N_{31}, the angular velocity ratio, are given:

Construction: (1) Lay down axes 12 and 13 making with each other the given angle α_1; (2) on axis 13 lay off OQ to scale proportional to N_{31}; (3) through point Q draw a line parallel to axis 12, and along it lay off QP to the same scale proportional to N_{21}, thus locating point P. OP will be velocity axis 23 and the angles into which it divides the angle between 12 and 13 will be α_2 and α_3.

Notes: (1) In making this construction the angular velocity scale ought of course, to be so chosen as to throw point P at a reasonably great distance from point O. (2) The student must understand that this graphical construction would not give sufficiently accurate values for α_2 and α_3 for shop purposes. Formulas for computing these angles with the necessary degree of accuracy will be derived a little later.

(b) *The Expression for the Angular Velocity Ratio* N_{12}/N_{32}.--For this ratio, as the symbolism shows, link 2 is the link of reference and links 1 and 3 the moving links. In Fig. 8-5 imagine cone 2 to be held stationary. Then baselink 1 rotates relative to link 2 about fixed velocity-axis 12, and cone 3 rotates at the instant under consideration about instantaneous velocity-axis 23 relative to link 2. Now consider the velocities of the two coincident points S_1 and S_3, *anywhere* on velocity-axis 13. The velocity of point S_1, a point in link 1, relative to link 2 is

$$S_1V_{12} = 2\pi HS_1N_{12}, \quad \ldots \ldots \ldots \ldots \quad (10)$$

in which HS_1 is the perpendicular distance from point S_1 to velocity-axis 12, the axis about which link 1 is rotating relative to the link of reference, link 2. Likewise the velocity of coincident point S_3 in link 3 is

$$S_3V_{32} = 2\pi TS_3N_{32} \quad \ldots \ldots \ldots \ldots \quad (11)$$

in which TS_3 is the perpendicular distance from point S_3 to velocity-axis 23, the axis about which link 3 is rotating relative to the link of reference, link 2. Now points S_1 and S_3 have no relative velocity. They must therefore have the same velocity relative to any given body of reference--in this case body 2. Thus

$$S_1V_{12} = S_3V_{32}, \quad \ldots \ldots \ldots \ldots \quad (12)$$

and

$$2\pi HS_1N_{12} = 2\pi TS_3N_{32}, \quad \ldots \ldots \ldots \ldots \quad (13)$$

or

$$\frac{N_{12}}{N_{32}} = \frac{TS_3}{HS_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

Now divide TS_3 and HS_1 by the common hypotenuse of right-angle triangles OHS_1 and OTS_1, and we have

$$\frac{N_{12}}{N_{32}} = \frac{\frac{TS_3}{OS_1}}{\frac{HS_1}{OS_1}} = \frac{\sin \alpha_3}{\sin \alpha_1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

an expression of exactly the same form as that derived for angular velocity ratio N_{21}/N_{31}, equation (7).

Applying the law of sines again to triangle OPQ there results the expression

$$\frac{\sin \alpha_3}{\sin \alpha_1} = \frac{QP}{OP}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

Thus, equation (15),

$$\frac{N_{12}}{N_{32}} = \frac{QP}{OP}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

from which it follows that to whatever scale QP represents angular velocity N_{12}, OP to the same scale represents angular velocity N_{32}.

(c) *The Expression for Angular Velocity Ratio* N_{13}/N_{23}.--Proceeding just as we did in deriving equations (14), (15), (16), and (17), using perpendiculars Q_2F and Q_2N from the two coincident points Q_1 and Q_2, anywhere on velocity-axis 12, to velocity-axes 13 and 23, it is easy to derive the corresponding expressions for N_{13}/N_{23}

$$\frac{N_{13}}{N_{23}} = \frac{Q_2N}{Q_1F}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

$$= \frac{\sin \alpha_2}{\sin \alpha_1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

$$= \frac{OQ}{OP}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

From equation (20) it is evident that to whatever scale OP represents angular velocity N_{23}, OQ to the same scale represents angular velocity N_{13}.

Thus it follows from *any* two of the three equations (9), (17), (20) that to whatever angular-velocity scale side OQ of triangle OQP represents $N_{31}(= -N_{13})$, the other two sides, QP and OP, represent, respectively, $N_{21}(= -N_{12})$ and $N_{23}(= -N_{32})$. Hence the three angular velocities N_{21}, N_{31}, and N_{32} are connected by the vector equation

$$N_{31} = N_{21} + N_{32}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

because this equation is merely the symbolic statement of what we have just proved, namely, that if we draw a triangle with its sides parallel, respectively, to the three velocity-axes 12, 13, 23, the sides of this triangle will be proportional in length to the magnitudes of the corresponding angular velocities.

To make equations (7), (15), and (19) fully useful it is necessary to be able to tell in any case whether the two moving bodies or links for any one of the three ratios are turning in the same sense or in opposite senses relative to the third of the three bodies involved in the ratio. This we do by a convention of signs.

Convention of Signs for Angular Velocity Ratios N_{21}/N_{31}, N_{12}/N_{32}, N_{13}/N_{23}.--In the ratio N_{21}/N_{31} link 1 is the link of reference and links 2 and 3 the moving links. In ratio N_{12}/N_{32} link 2 is the link of reference and links 1 and 3 the moving links. In ratio N_{13}/N_{23} link 3 is the link of reference and links 1 and 2 the moving links. In any one of the ratios the subscript of the numerator designates the primary velocity-axis, the subscript of the denominator the secondary velocity-axis, and the symbol derived by combining the first numerals of the two subscripts designates the mutual velocity-axis. For the ratio N_{21}/N_{31}, for example, 21 designates the primary axis, 31 the secondary axis, and 23 the mutual axis.

Rule of Signs.--With the eye at O, the point of intersection of the three velocity-axes, and limiting each of the angles α_1, α_2, and α_3 to a maximum of 180°, make the sign of any one of the three angular velocity ratios positive if the two moving links of the ratio turn in the same sense relative to the link of reference; make the sign negative if they turn in opposite senses relative to the link of reference.

If the mutual velocity-axis of a ratio lies between the other two velocity-axes the moving links obviously turn in opposite senses relative to the link of reference. If, on the other hand, the mutual velocity-axis for a ratio does not lie between the other two velocity-axes the moving links turn in the same sense relative to the link of reference.

The Rule of Signs may therefore be stated as follows: Make the sign of any one of the three ratios *negative* if its mutual velocity-axis lies between the other two axes; make the sign *positive* if the mutual velocity-axis does not lie between the other two axes.

Example: The sign of the ratio N_{21}/N_{31} for the case shown in Fig. 8-5 should be negative because the mutual velocity-axis 23 lies between the other two, and thus links 2 and 3 turn in opposite senses relative to link 1 when observed from point O. The sign of N_{12}/N_{32}, on the other hand, is positive because the mutual velocity-axis 13 does not lie between the other two; thus links 1 and 3 turn in the same sense relative to link 2 when observed from point O. For the same reason the sign of the ratio N_{13}/N_{23} is positive.

Problem 1.--Given N_{21}(clockwise), α_2, and α_3; to find α_1, N_{31}, and N_{23}.

Solution.--If α_3 is greater than α_2 two solutions arise according as

$$\alpha_1 = \alpha_2 + \alpha_3, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1_1)$$

or

$$\alpha_1 = \alpha_3 - \alpha_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2_1)$$

In the first case we have from equation (7), p. 110,

$$N_{31} = -\frac{\sin \alpha_2}{\sin \alpha_3} N_{21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3_1)$$

and in the second

$$N_{31} = \frac{\sin \alpha_2}{\sin \alpha_3} N_{21}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4_1)$$

For N_{23} we have, equation (19), p. 112,

$$N_{23} = \frac{\sin(\alpha_2 + \alpha_3)}{\sin \alpha_2} N_{13}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5_1)$$

or

$$N_{23} = -\frac{\sin(\alpha_3 - \alpha_2)}{\sin \alpha_2} N_{13}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (6_1)$$

according as we make use of assumption (1_1) or (2_1).

Moreover, equation (21), p. 112, gives us

$$N_{23} = N_{13} \leftrightarrow N_{21}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (7_1)$$

in which N_{13} is known from equation (3_1).

Problem 2.--Given α_1, Z_2, Z_3, and P_{23}; required α_2, α_3, R_2, and R_3. This is the kinematic problem that presents itself to the designer of a pair of meshing bevel gears.

Four cases arise. Case 1: $\alpha_1 = (\alpha_2 + \alpha_3) < 90^o$. Case 2: (1) $\alpha_1 = (\alpha_2 + \alpha_3) > 90^o$; (2) $\alpha_2 < \alpha_3 < 90^o$. Case 3: (1) $\alpha_1 = (\alpha_2 + \alpha_3) > 90^o$; (2) $\alpha_2 < \alpha_3 > 90^o$. Case 4: (1) $\alpha_1 = (\alpha_2 + \alpha_3) > 90^o$; (2) $\alpha_2 < \alpha_3 = 90^o$.

Solution.--Case 1: $\alpha_1 = (\alpha_2 + \alpha_3) < 90^o$. For this case Fig. 8-5 is suited. Our problem, mathematically, is to express the unknown quantities in terms of the known quantities.

(1) To find R_2 and R_3, the radii of the pitch-cones at their large ends, we have

$$R_2 = \frac{Z_2}{2P_{23}} \ldots (2_2) \text{ and } R_3 = \frac{Z_3}{2P_{23}} \ldots (2_2)$$

just as for spur-gears.

(2) To express α_2, the half pitch-cone angle of gear 2, in terms of α_1 Z_2 and Z_3 draw Q_2F perpendicular and P_2E parallel to velocity-axis 13. Then from triangle OQ_2P_2

$$\tan \alpha_2 = \frac{P_2Q_2}{Q_2O} = \frac{R_2}{Q_2O}. \quad \ldots\ldots\ldots\ldots\ldots (3_2)$$

But from triangle OQ_2F

$$Q_2O = \frac{Q_2F}{\sin \alpha_1}, \quad \ldots\ldots\ldots\ldots\ldots (4_2)$$

therefore, equations (3_2) and (4_2),

$$\tan \alpha_2 = \frac{R_2 \sin \alpha_1}{Q_2F}. \quad \ldots\ldots\ldots\ldots\ldots (5_2)$$

From the figure:

$$Q_2F = Q_2E + EF, \quad \ldots\ldots\ldots\ldots\ldots (6_2)$$

$$Q_2E = R_2 \cos \alpha, \quad \ldots\ldots\ldots\ldots\ldots (7_2)$$

and

$$EF = P_3S_3 = R_3. \quad \ldots\ldots\ldots\ldots\ldots (8_2)$$

Hence, equations (5_2), (6_2), (7_2), (8_2),

$$\tan \alpha_2 = \frac{R_2 \sin \alpha_1}{R_3 + R_2 \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9_2)$$

$$= \frac{\sin \alpha_1}{\frac{R_3}{R_2} + \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10_2)$$

Now from equation (6), p. 110

$$\frac{R_3}{R_2} = \frac{Z_3}{Z_2}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11_2)$$

consequently equation (10_2) becomes

$$\tan \alpha_2 = \frac{\sin \alpha_1}{\frac{Z_3}{Z_2} + \cos \alpha_1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12_2)$$

$$= \frac{\sin \alpha_1}{\frac{N_{21}}{N_{31}} + \cos \alpha_1}.$$

(3) To determine α_3 for this case we can use exactly the same method we used in determining α_2, substituting corresponding construction lines S_3H and P_3G for construction lines Q_2F and P_2E. Thus we obtain for the half pitch-cone angle of gear 3 the expression

$$\tan \alpha_3 = \frac{\sin \alpha_1}{\frac{Z_2}{Z_3} + \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13_2)$$

$$= \frac{\sin \alpha_1}{\frac{N_{31}}{N_{21}} + \cos \alpha_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14_2)$$

Or, since by hypothesis

$$\alpha_1 = \alpha_2 + \alpha_3, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15_2)$$

we can find α_3 when α_2 has been computed from the simple formula

$$\alpha_3 = \alpha_1 - \alpha_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16_2)$$

Case 2: (1) $\alpha_1 = (\alpha_2 + \alpha_3) > 90^\circ$; (2) $\alpha_2 < \alpha_3 < 90^\circ$. For this case we need a new figure, Fig. 8-6, which is drawn and symbolized to correspond to Fig. 8-5.

(1) To find R_2 and R_3 we use formulas (1_2) and (2_2), Case 1.

(2) To determine α_2 we have, as for Case 1, from triangle OQ_2P_2

$$\tan \alpha_2 = \frac{P_2Q_2}{Q_2O} = \frac{R_2}{Q_2O}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17_2)$$

But from triangle OQ_2F

$$Q_2O = \frac{Q_2F}{\sin(180 - \alpha_1)} = \frac{Q_2F}{\sin \alpha_1}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18_2)$$

therefore, equations (17_2) and (18_2)

$$\tan \alpha_2 = \frac{R_2 \sin \alpha_1}{Q_2F} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19_2)$$

From the figure, moreover,

$$Q_2F = FE - Q_2E, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20_2)$$

$$FE = P_3S_3 = R_3, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21_2)$$

and

$$Q_2E = R_2 \cos(180 - \alpha_1), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22_2)$$

$$= -R_2 \cos \alpha_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23_2)$$

Therefore, equations (19_2), (20_2), (21_2), (22_2), (23_2),

$$\tan \alpha_2 = \frac{R_2 \sin \alpha_1}{R_3 + R_2 \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24_2)$$

$$= \frac{\sin \alpha_1}{\frac{R_3}{R_2} + \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25_2)$$

$$= \frac{\sin \alpha_1}{\frac{Z_3}{Z_2} + \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (26_2)$$

$$= \frac{\sin \alpha_1}{\frac{N_{21}}{N_{31}} + \cos \alpha_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (27_2)$$

Formulas (25_2), (26_2), and (27_2) for this case, in which α_1 is greater than 90°, are exactly like formulas (10_2), (12_2), and (14_2) for the case in which α_1 is less than 90°. But the student must remember, in using formulas (25_2), (26_2), and (27_2), that since α_1 is greater than 90° $\cos \alpha_1$ is negative.

(3) To determine α_3, the half pitch-cone angle of gear 3, for this case the procedure is exactly the same as that used in finding α_2, but, of course, we must now use construction lines GH and P_3G instead of construction lines EF and P_2E. Thus we find

$$\tan \alpha_3 = \frac{\sin \alpha_1}{\frac{R_2}{R_3} + \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (28_2)$$

$$= \frac{\sin \alpha_1}{\frac{Z_2}{Z_3} + \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29_2)$$

$$= \frac{\sin \alpha_1}{\frac{N_{31}}{N_{21}} + \cos \alpha_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (30_2)$$

And here again it must be remembered that since α_1 is greater than 90^o, cos α_1 is negative in these three formulas for tan α_3.

Obviously, moreover, for this case as for Case 1,

$$\alpha_3 = \alpha_1 - \alpha_2 ,$$

from which when α_2 has been computed α_3 can be found.

Case 3: (1) $\alpha_1 = \alpha_2 + \alpha_3 > 90^o$; (2) $\alpha_2 < \alpha_3 > 90^o$. For this case also, the case of the internal bevel-gear, we need a different figure; Fig. 8-7, which is symbolized to correspond exactly to Figs. 8-5 and 8-6.

(1) For this case, just as for the other two cases, R_2 and R_3 are found from the equations (1_2) and (2_2), Case 1.

$$R_2 = \frac{Z_2}{2P_{23}} \quad . \quad . \quad . \quad (31_2) \text{ and } R_3 = \frac{Z_3}{2P_{23}} \quad . \quad . \quad . \quad (32_2)$$

(2) And to find α_2 the procedure is precisely the same as that used in finding α_2 for Cases 1 and 2.

From triangle OQ_2P_2

$$\tan \alpha_2 = \frac{P_2Q_2}{Q_2O} = \frac{R_2}{Q_2O} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33_2)$$

But from triangle OQ_2F

$$Q_2O = \frac{Q_2F}{\sin(180 - \alpha_1)} = \frac{Q_2F}{\sin \alpha_1} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34_2)$$

therefore, equations (33_2) and (34_2)

$$\tan \alpha_2 = \frac{R_2 \sin \alpha_1}{Q_2F} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35_2)$$

From the figure we have, moreover,

$$Q_2F = EF - Q_2E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36_2)$$

$$Q_2E = R_2 \cos(180 - \alpha_1) = -R_2 \cos \alpha_1 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37_2)$$

and

$$EF = P_3S_3 = R_3 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38_2)$$

Thus, equations (35_2), (36_2), (37_2), (38_2),

$$\tan \alpha_2 = \frac{R_2 \sin \alpha_1}{R_3 + R_2 \cos \alpha_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39_2)$$

$$= \frac{\sin \alpha_1}{\frac{R_3}{R_2} + \cos \alpha_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40_2)$$

$$\tan \alpha_2 = \frac{\sin \alpha_1}{\frac{Z_3}{Z_2} + \cos \alpha_1} \quad \ldots\ldots\ldots\ldots \quad (41_2)$$

$$= \frac{\sin \alpha_1}{\frac{N_{21}}{N_{31}} + \cos \alpha_1}. \quad \ldots\ldots\ldots\ldots \quad (42_2)$$

(3) Similarly, using construction lines S_3H and P_3G instead of construction lines Q_2F and P_2E, we have

$$\tan \alpha_3 = \frac{\sin \alpha_1}{\frac{R_2}{R_3} + \cos \alpha_1} \quad \ldots\ldots\ldots\ldots \quad (43_2)$$

$$= \frac{\sin \alpha_1}{\frac{Z_2}{Z_3} + \cos \alpha_1} \quad \ldots\ldots\ldots\ldots \quad (44_2)$$

$$= \frac{\sin \alpha_1}{\frac{N_{31}}{N_{21}} + \cos \alpha_1}. \quad \ldots\ldots\ldots\ldots \quad (45_2)$$

Here also, since by hypothesis

$$\alpha_1 = \alpha_2 + \alpha_3 ,$$

we have

$$\alpha_3 = \alpha_1 - \alpha_2 \quad \ldots\ldots\ldots\ldots \quad (46_2)$$

with which α_3, the half pitch-cone angle of the gear, can be computed when α_2, the half pitch-cone angle of the pinion, has been found.

Case 4: (1) $\alpha_1 = \alpha_2 + \alpha_3 > 90^o$; (2) $\alpha_2 < \alpha_3 = 90^o$. For this case, we have Fig. 8-8, which is symbolized to correspond with Figs. 8-5, 8-6, and 8-7.

Since here $\alpha_3 = 90^o$ the pitch cone of gear 3 becomes a plane and the gear becomes what is called a crown gear. The crown gear in the bevel-gear system corresponds kinematically in many ways very closely to the rack in the spur-gear system.

(1) To determine R_2 and R_3 formulas (1_2), (2_2) of Case 1 are used.
(2) To determine α_2 for this case we have as in the three preceding cases

$$\tan \alpha_2 = \frac{P_2Q_2}{Q_2O} = \frac{R_2}{Q_2O}. \quad \ldots\ldots\ldots\ldots \quad (47_2)$$

But from triangle OQ_2F

$$Q_2O = \frac{Q_2F}{\sin(180 - \alpha_1)} = \frac{Q_2F}{\sin \alpha_1}, \quad \ldots\ldots\ldots\ldots \quad (48_2)$$

therefore, equations (47_2) and (48_2),

$$\tan \alpha_2 = \frac{R_2 \sin \alpha_1}{Q_2F}. \quad \ldots\ldots\ldots\ldots \quad (49_2)$$

From the figure

$$O_2F = FE - O_2E, \quad \ldots \quad (50_2)$$

$$FE = P_3S_3 = R_3, \quad \ldots \quad (51_2)$$

$$O_2E = R_2 \cos(180 - \alpha_1) \quad \ldots \quad (52_2)$$

$$= -R_2 \cos \alpha_1. \quad \ldots \quad (53_2)$$

Therefore, equations (49_2)-(53_2),

$$\tan \alpha_1 = \frac{R_2 \sin \alpha_1}{R_3 + R_2 \cos \alpha_1}. \quad \ldots \quad (54_2)$$

$$= \frac{\sin \alpha_1}{\frac{R_3}{R_2} + \cos \alpha_1}, \quad \ldots \quad (55_2)$$

$$= \frac{\sin \alpha_1}{\frac{Z_3}{Z_2} + \cos \alpha_1}, \quad \ldots \quad (56_2)$$

$$= \frac{\sin \alpha_1}{\frac{N_{21}}{N_{31}} + \cos \alpha_1}. \quad \ldots \quad (57_2)$$

(3) For this case no formula for α_3 is necessary since its value is given.

Comments on the Four Cases.--(1) in all four cases the expressions for tan α_2 are exactly alike in form and so are the expressions for tan α_3. But in using the formulas it is necessary, of course, to make cos α_1 negative whenever α_1 is greater than 90^o.

(2) For Case 3, Fig. 8-7, if α_1 designates the *acute* angle between axes 12 and 13 and α_3 designates the acute angle between axes 23 and 13, that is, the half pitch-cone angle of gear 3, then it is easy to see that

$$\tan \alpha_2 = \frac{\sin \alpha_1}{\frac{R_3}{R_2} - \cos \alpha_1}, \quad \ldots \quad (58_2)$$

$$= \frac{\sin \alpha_1}{\frac{Z_3}{Z_2} - \cos \alpha_1} \quad \ldots \quad (59_2)$$

$$= \frac{\sin \alpha_1}{\frac{N_{21}}{N_{31}} - \cos \alpha_1} \quad \ldots \quad (60_2)$$

and

$$\tan \alpha_3 = \frac{\sin \alpha_1}{\cos \alpha_1 - \frac{R_2}{R_3}} \quad \ldots \quad (61_2)$$

$$\tan \alpha_3 = \frac{\sin \alpha_1}{\cos \alpha_1 - \frac{Z_2}{Z_3}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (62_2)$$

$$= \frac{\sin \alpha_1}{\cos \alpha_1 - \frac{N_{31}}{N_{21}}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (63_2)$$

(3) If for Case 4, Fig. 8-8, α_1 designates the acute angle between axes 12 and 13 then it is obvious that

$$\tan \alpha_2 = \frac{\sin \alpha_1}{\frac{R_3}{R_2} - \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots \quad (64_2)$$

$$= \frac{\sin \alpha_1}{\frac{Z_3}{Z_2} - \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots \quad (65_2)$$

$$= \frac{\sin \alpha_1}{\frac{N_{21}}{N_{31}} - \cos \alpha_1} \quad \ldots\ldots\ldots\ldots\ldots \quad (66_2)$$

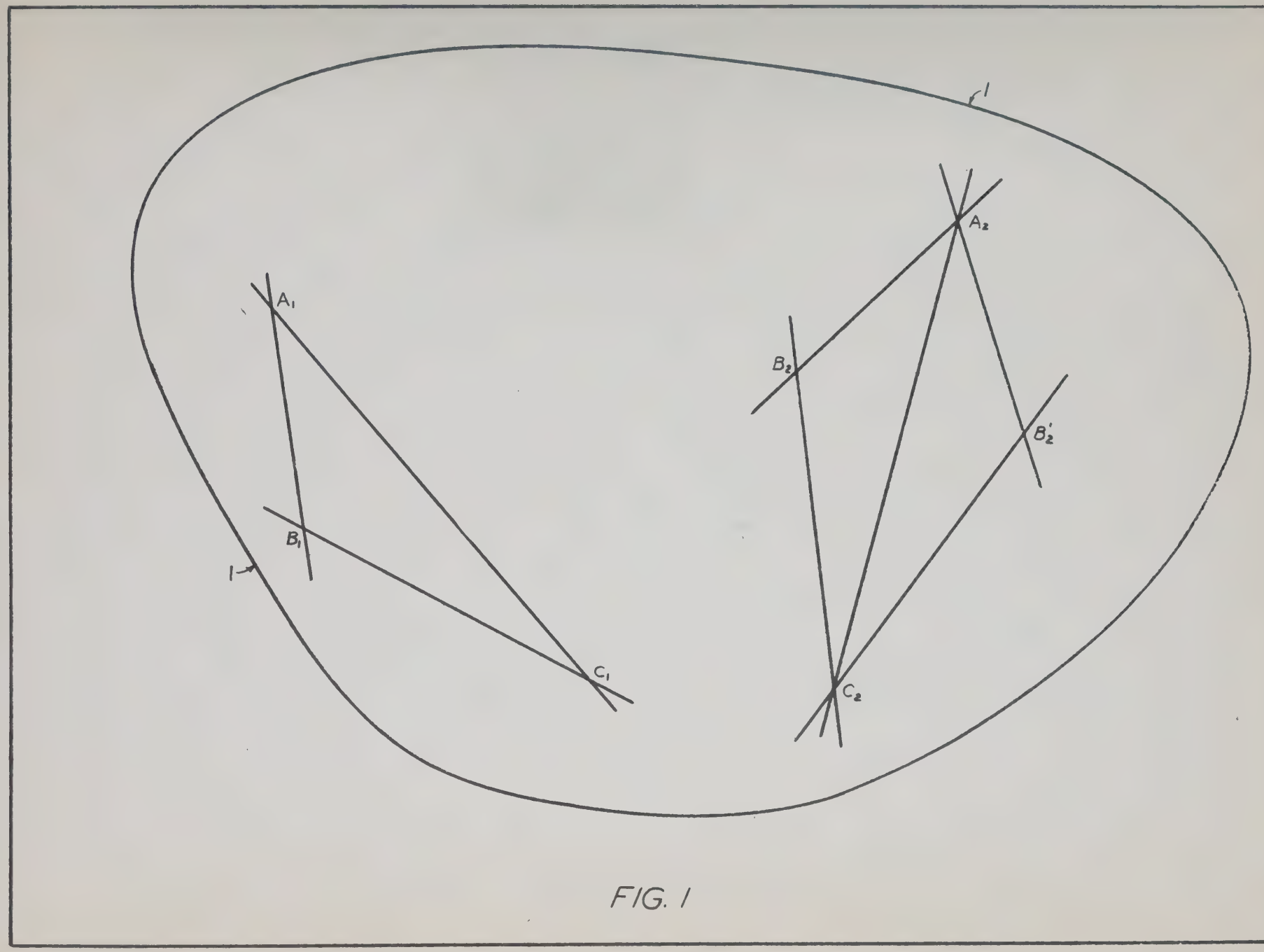


FIG. 1

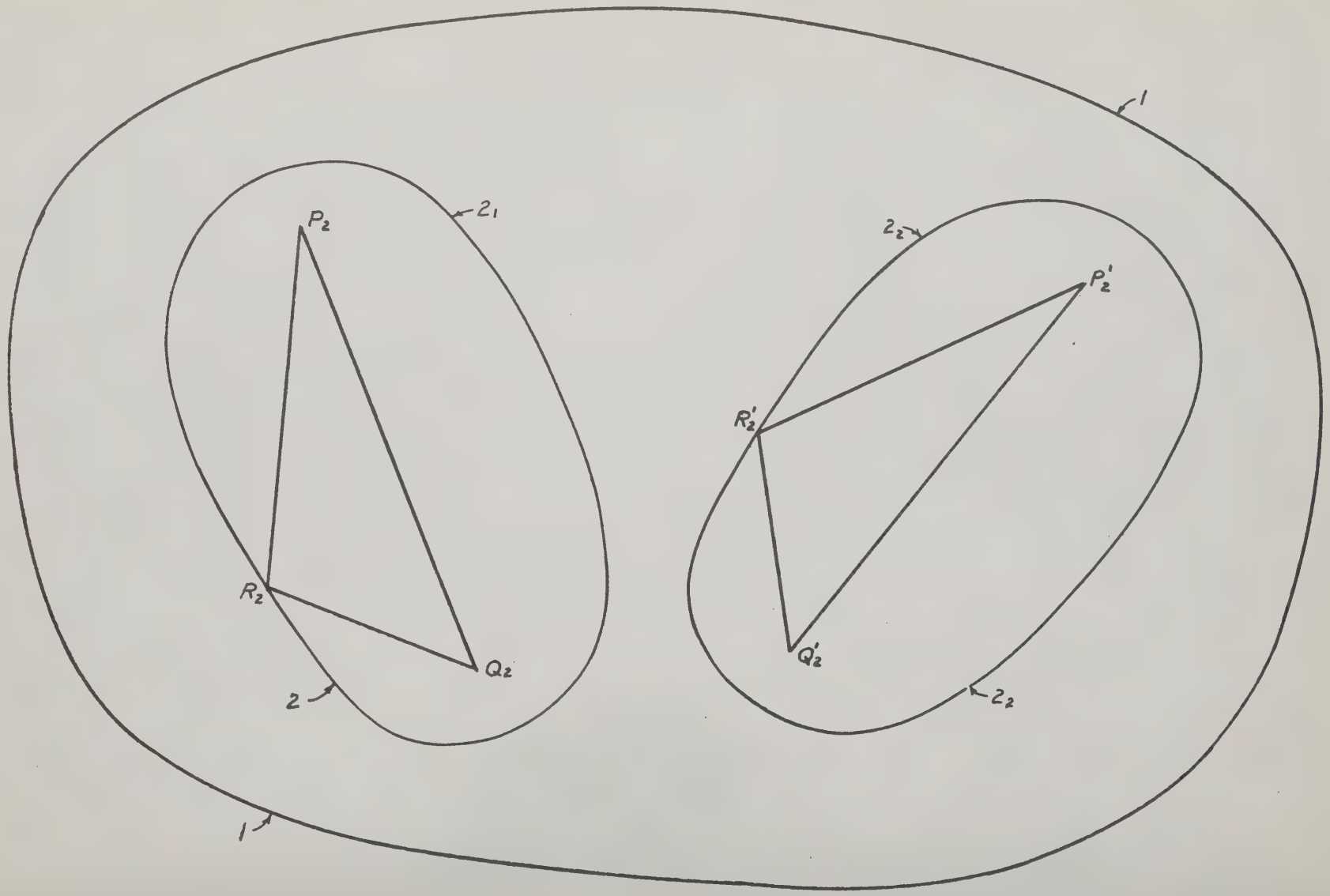


FIG. 2

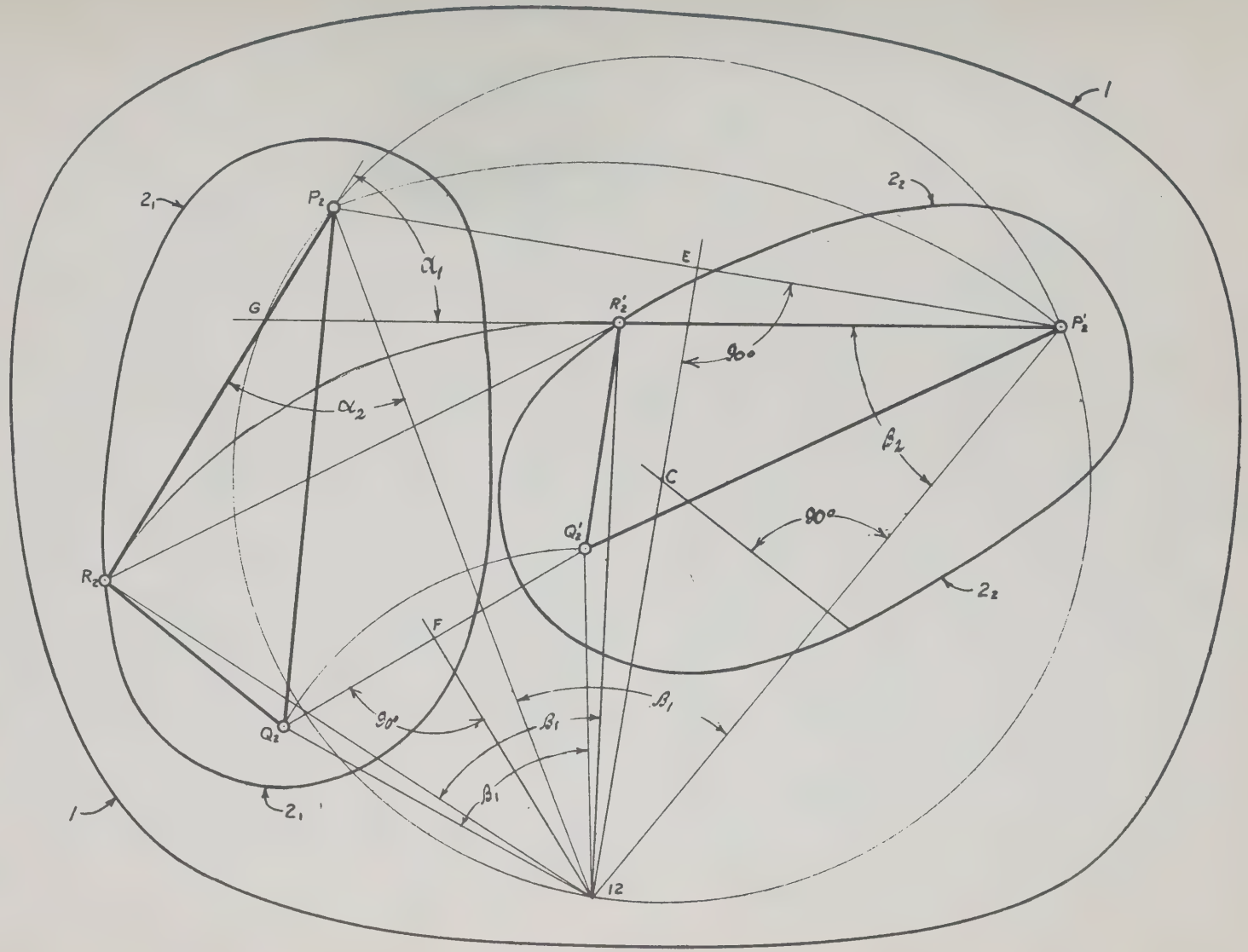


FIG. 3

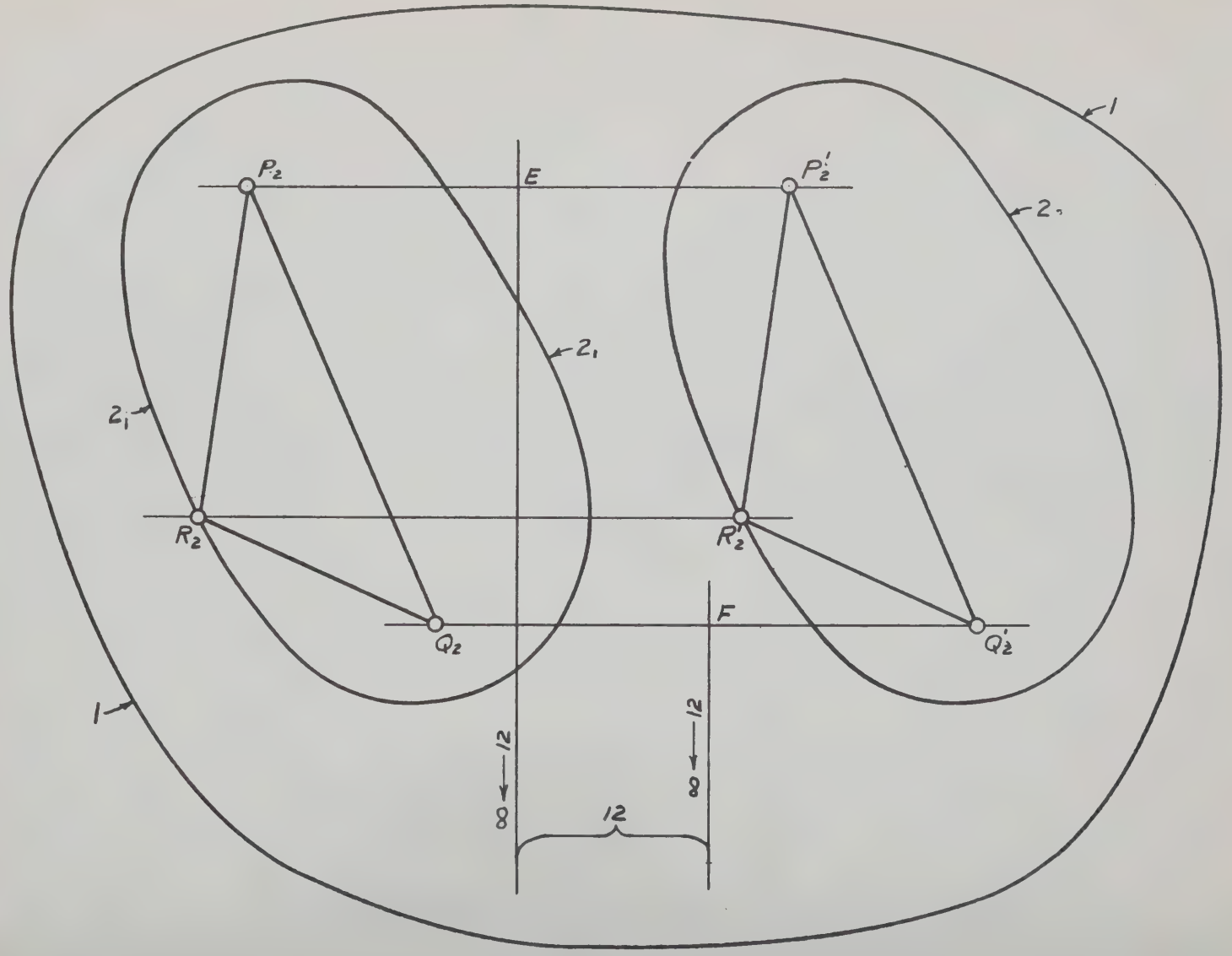


FIG. 4

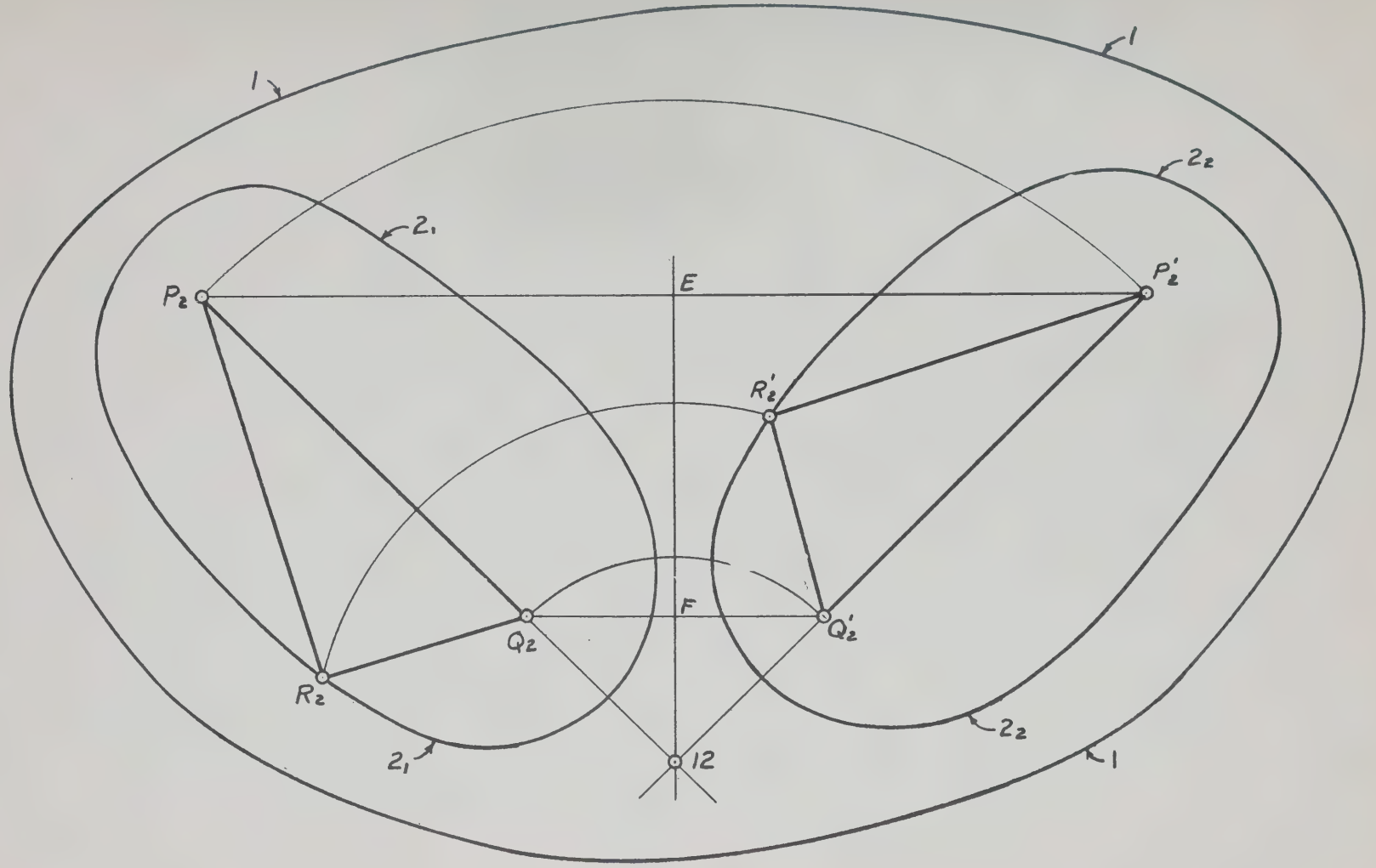


FIG. 5

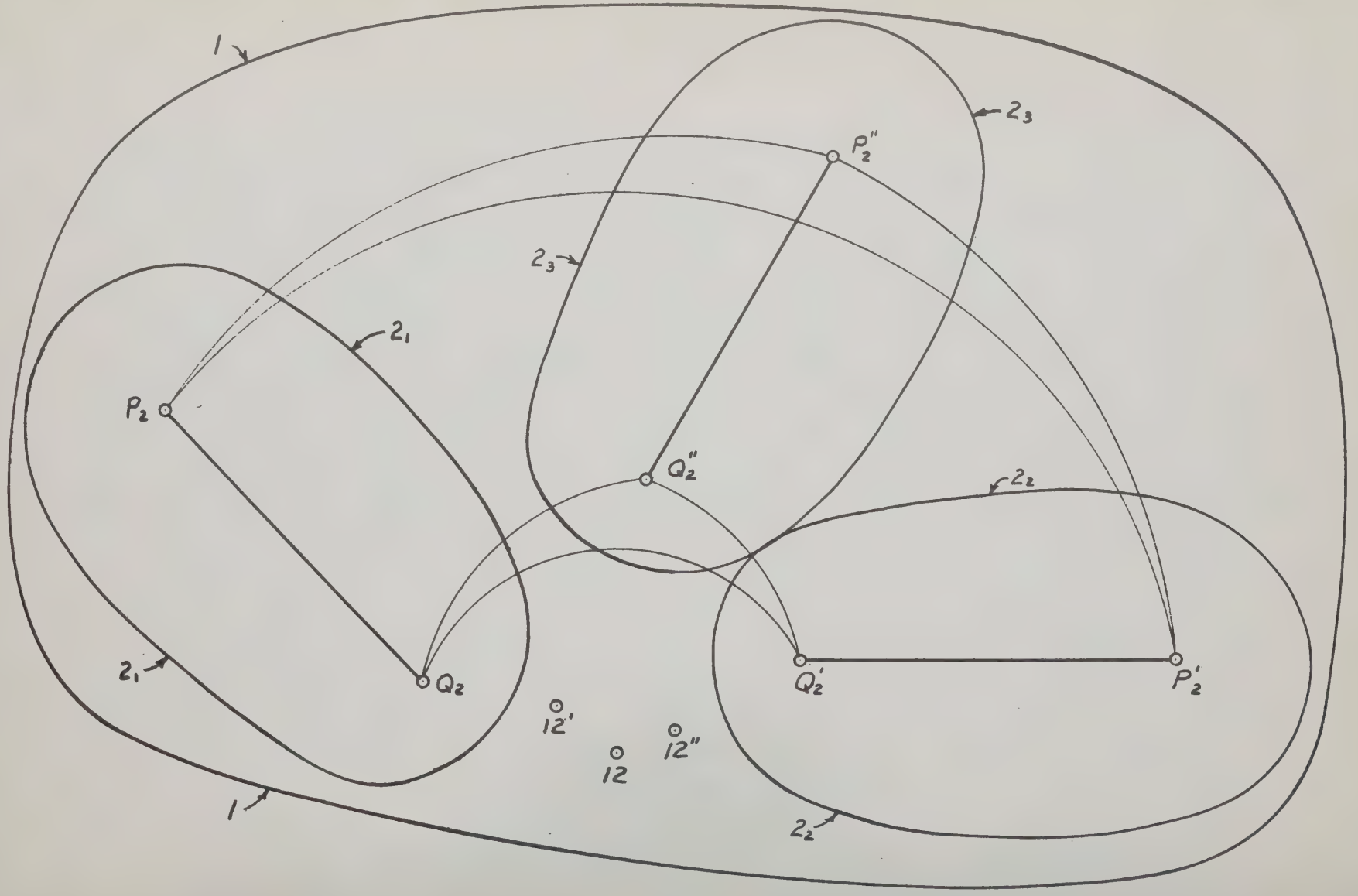


FIG. 6

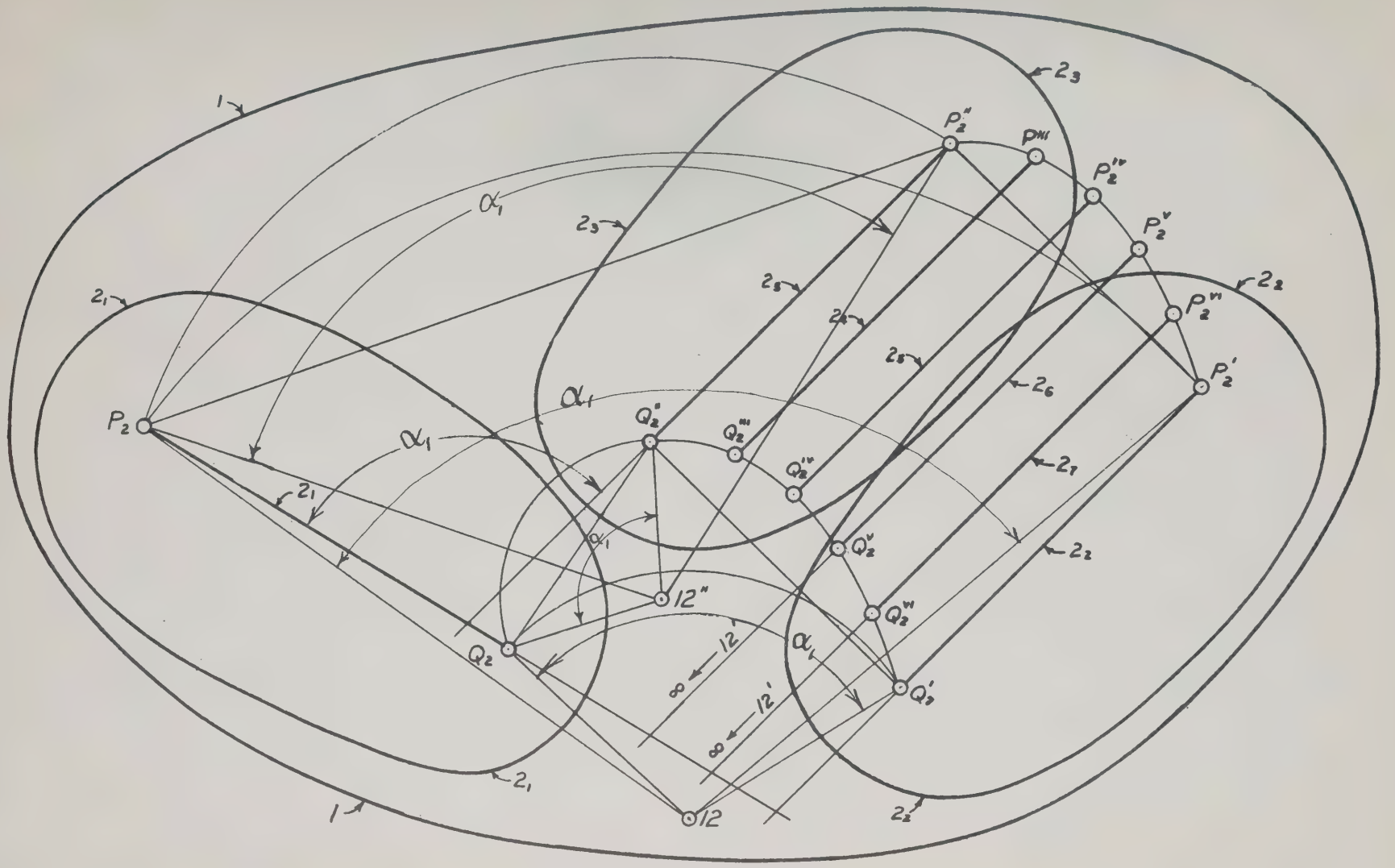

1
2₃
P₂''
P'''
P₂'''
P₂''
P₂'
2₂
2₄
2₅
2₆
2₇
P₂'
P₂''
2₁
α₁
P₂
Q₂'
Q₂'''
Q₂''
Q₂''
Q₂''
12''
Q₂
12'
12'
Q₂'
12
2₁
1
2₂

FIG. 7

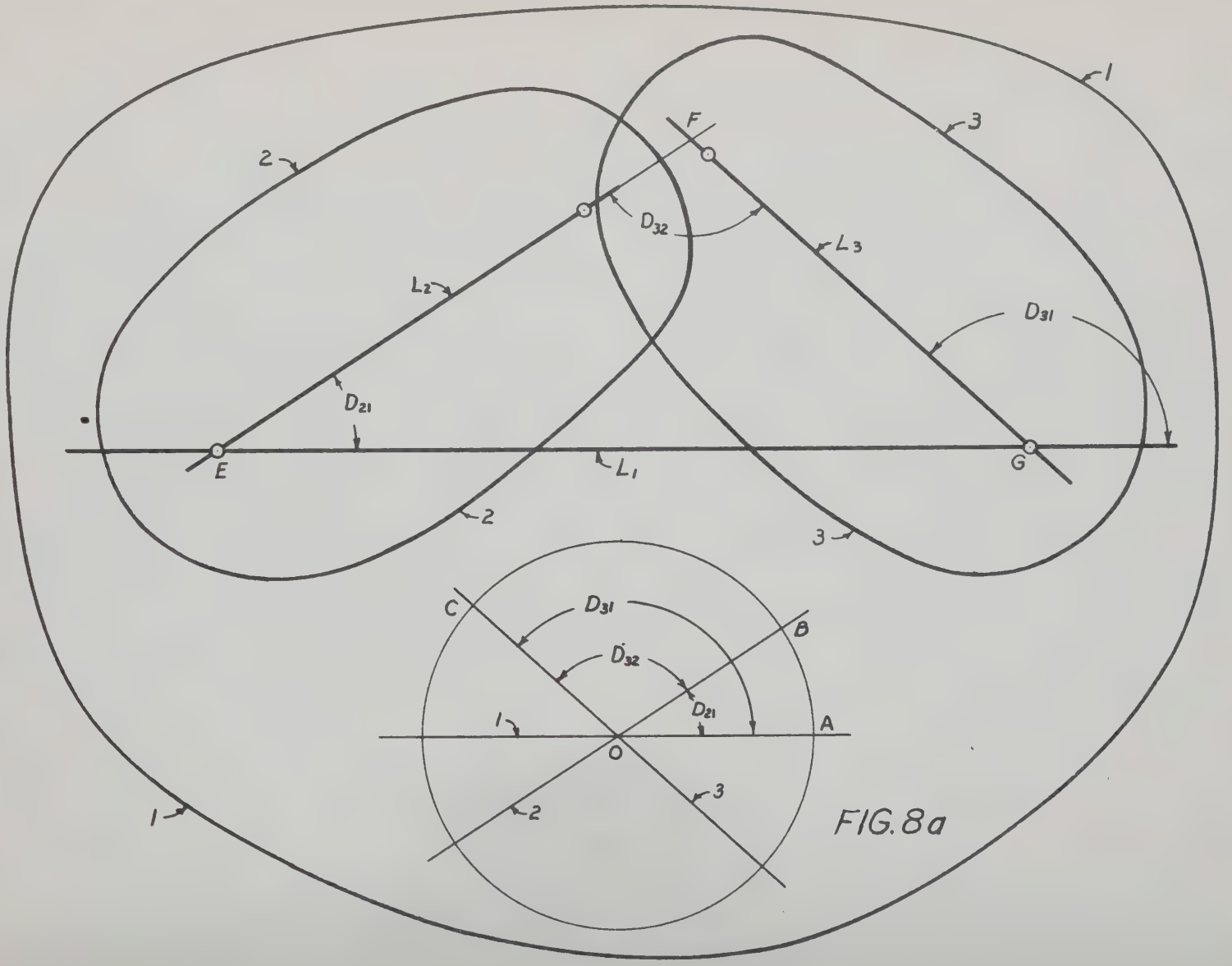


FIG. 8a

FIG. 8

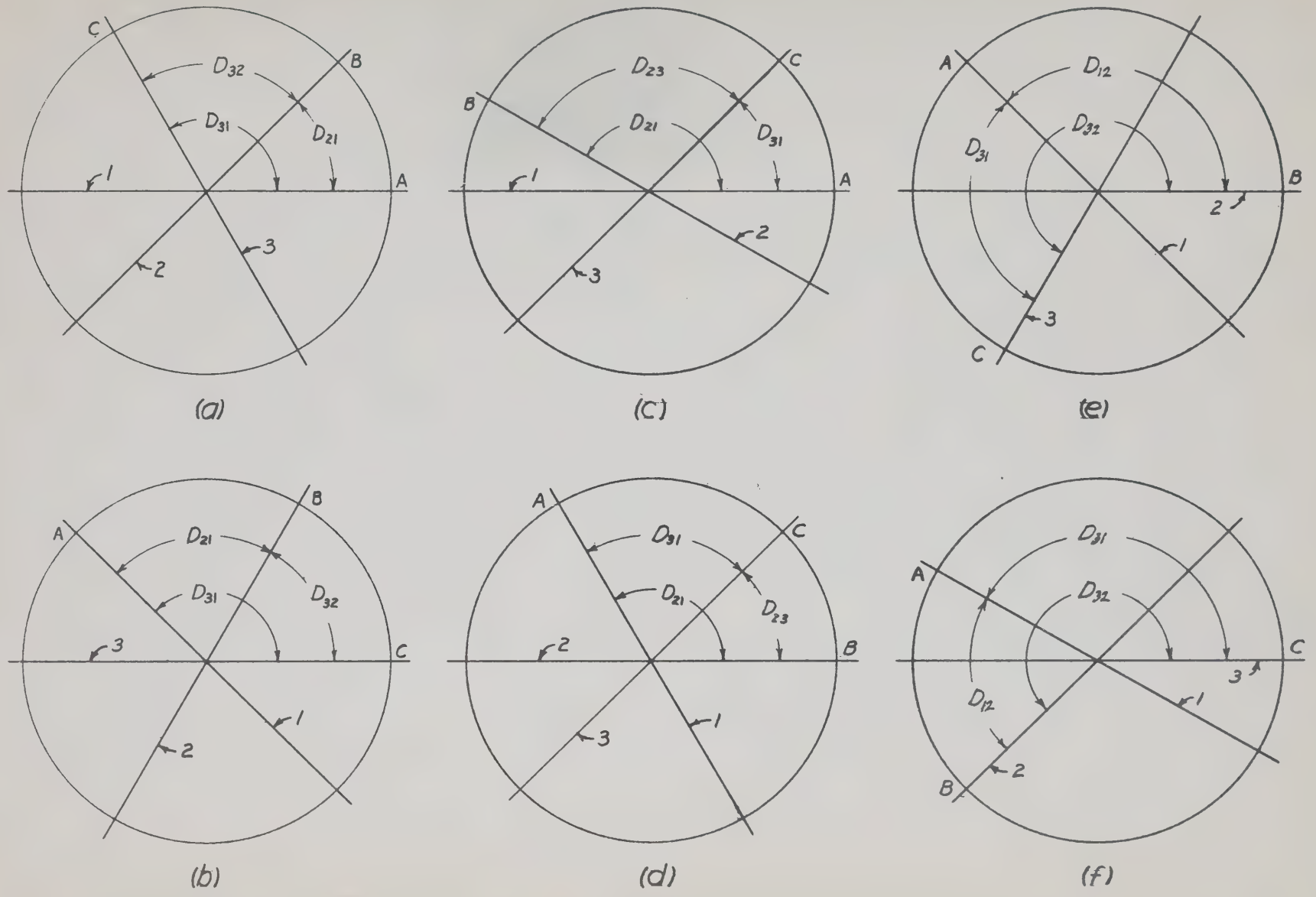

C
B
A
D_{32}
D_{31}
D_{21}
1
2
3
(a)
B
C
A
D_{23}
D_{21}
D_{31}
1
2
3
(c)
A
B
C
D_{12}
D_{32}
D_{31}
1
2
3
(e)
A
B
C
D_{21}
D_{31}
D_{32}
1
2
3
(b)
A
C
B
D_{31}
D_{21}
D_{23}
1
2
3
(d)
A
C
B
D_{31}
D_{32}
D_{12}
1
2
3
(f)

FIG 9

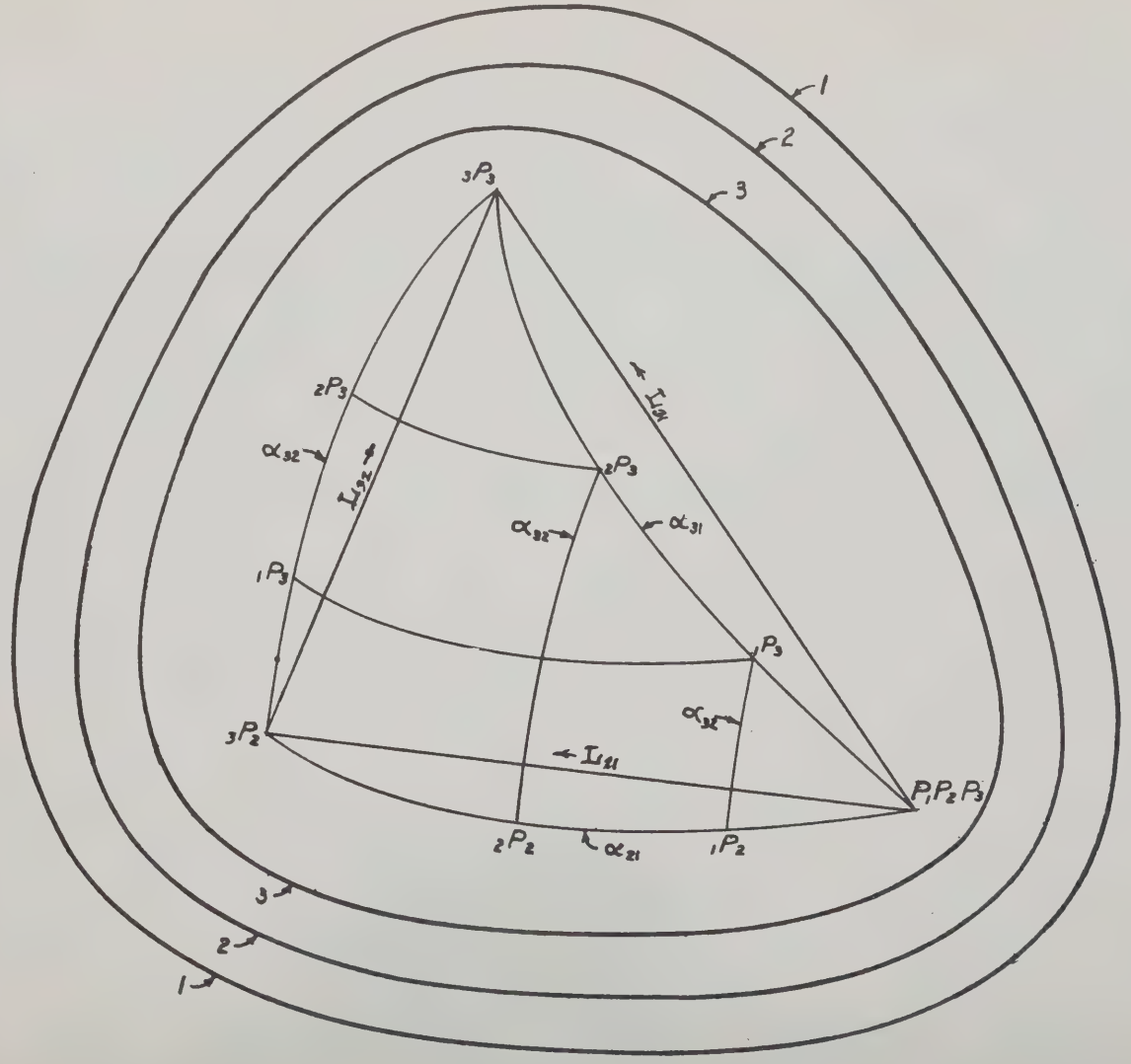

1
2
3
$_3P_3$
$_2P_3$
$_1P_3$
$_3P_2$
$_2P_2$
$_1P_2$
P_3
$P_1P_2P_3$
α_{32}
α_{31}
α_{21}
I_{32}
I_{31}
I_{21}
3
2
1

FIG. 10

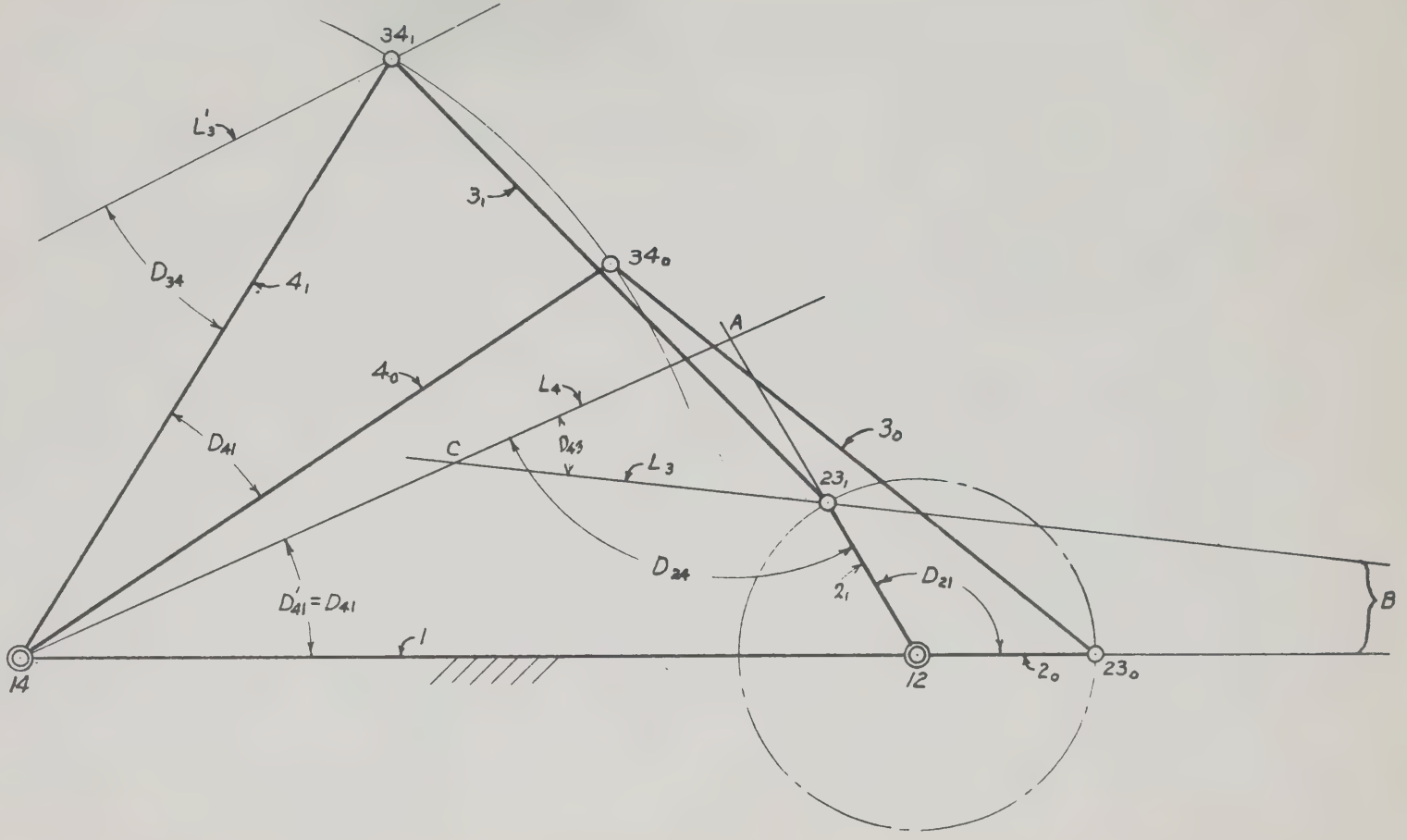

34_1
L_3'
3_1
D_{34}
4_1
34_0
A
4_0
L_4
3_0
C
D_{43}
L_3
23_1
D_{41}
D_{24}
D_{21}
2_1
B
$D_{41}' = D_{41}$
1
14
12
2_0
23_0

FIG. 11

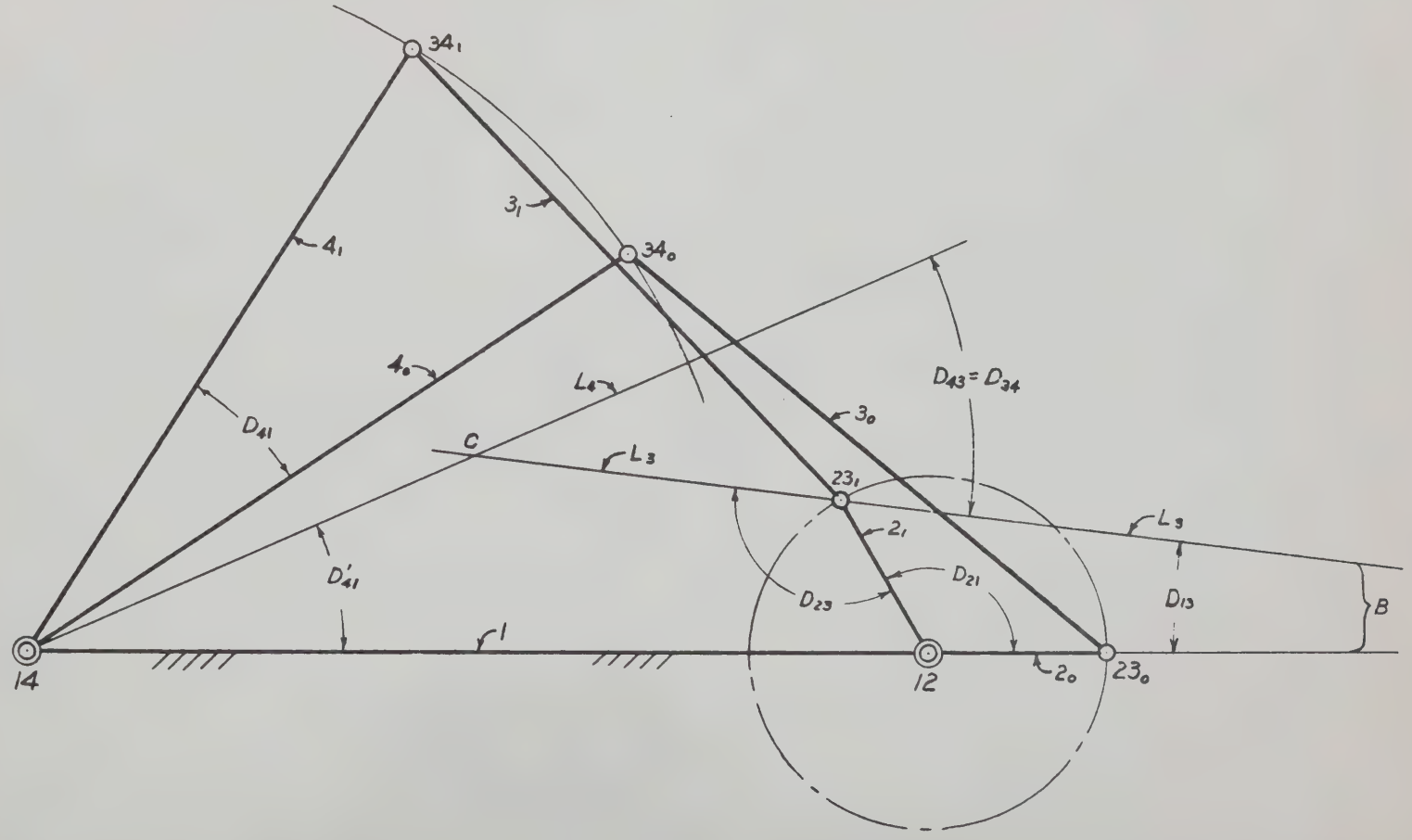

34₁
3₁
4₁
34₀
4₀
L₄
D₄₃ = D₃₄
3₀
D₄₁
C
L₃
23₁
L₂
2₁
D'₄₁
D₂₁
D₂₃
D₁₃
B
1
14
12
2₀
23₀

FIG. 12

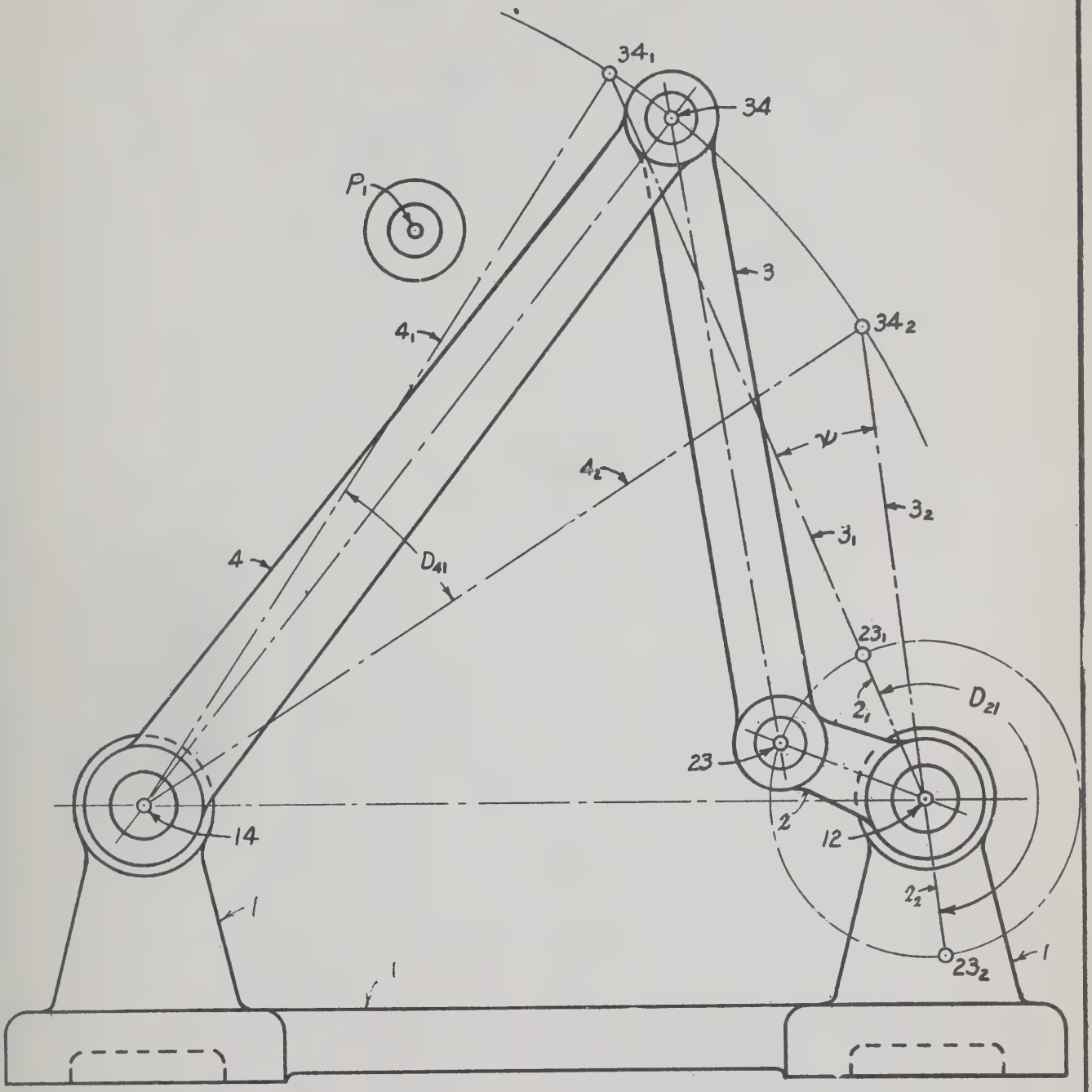


FIG. 13

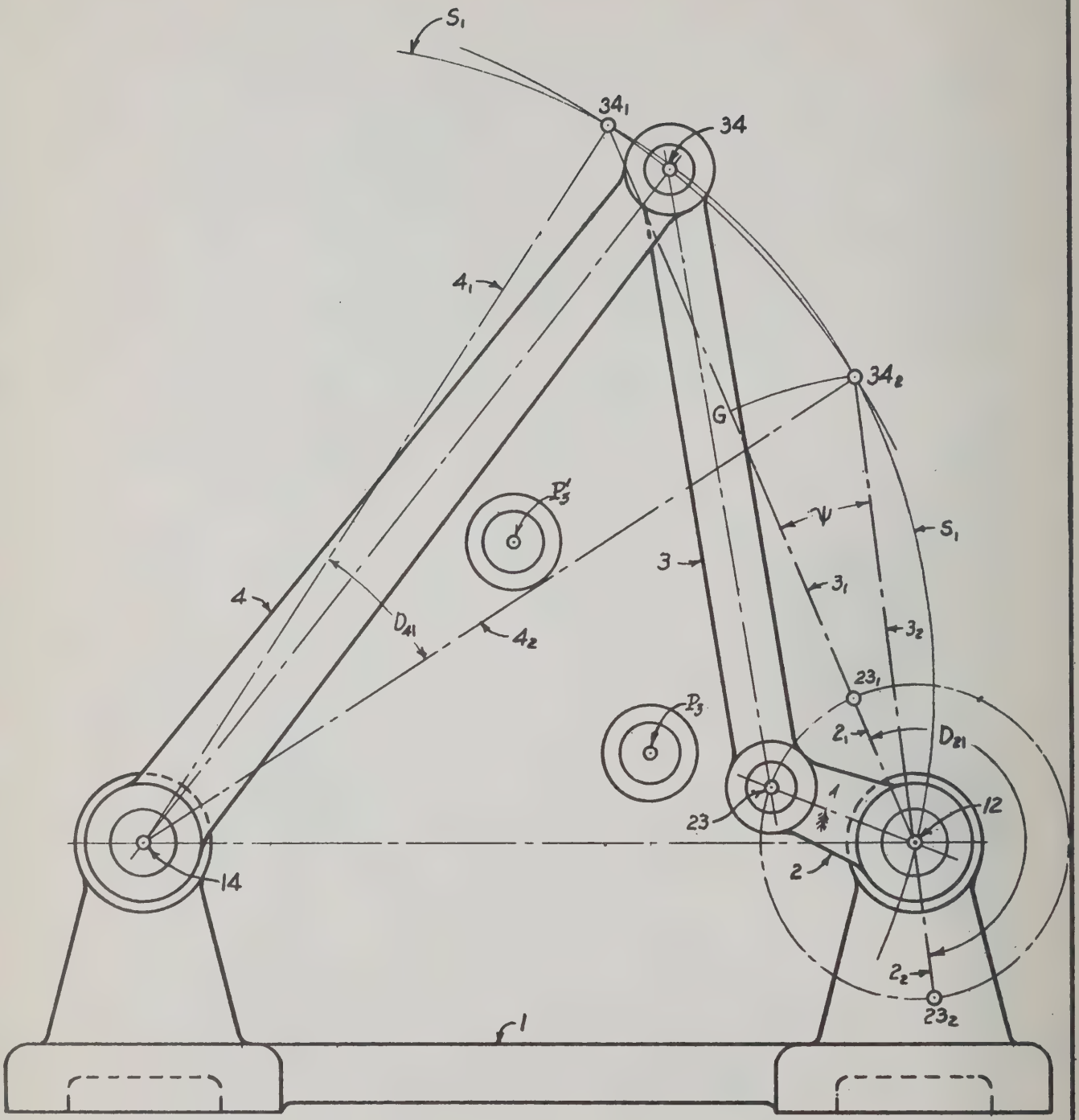

S1
34₁
34
4₁
34₂
G
P'3
ψ
S1
3
3₁
3₂
4
D41
4₂
23₁
P3
2₁
D21
23
1
12
14
2
2₂
23₂
1

FIG. 14

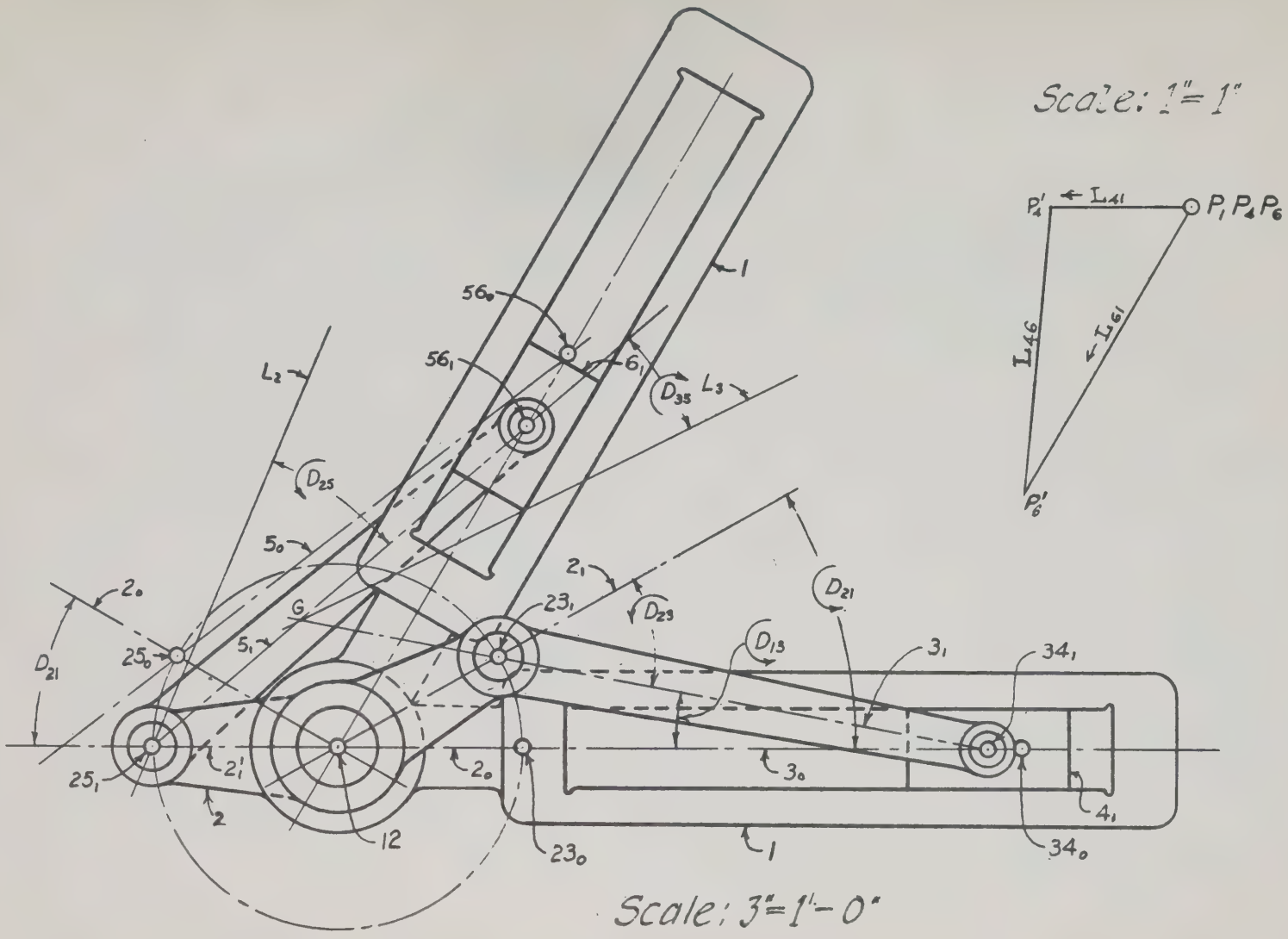

Scale: 1"=1'
P_4'
L_{41}
$P_1 P_4 P_6$
L_{46}
L_{61}
P_6'
56_0
56_1
6_1
D_{35}
L_3
L_2
D_{25}
5_0
2_0
G
5_1
25_0
D_{21}
25_1
$2_1'$
2
12
2_1
D_{23}
23_1
D_{21}
D_{13}
3_1
34_1
2_0
3_0
4_1
23_0
1
34_0
Scale: 3"=1'-0"

FIG. 15

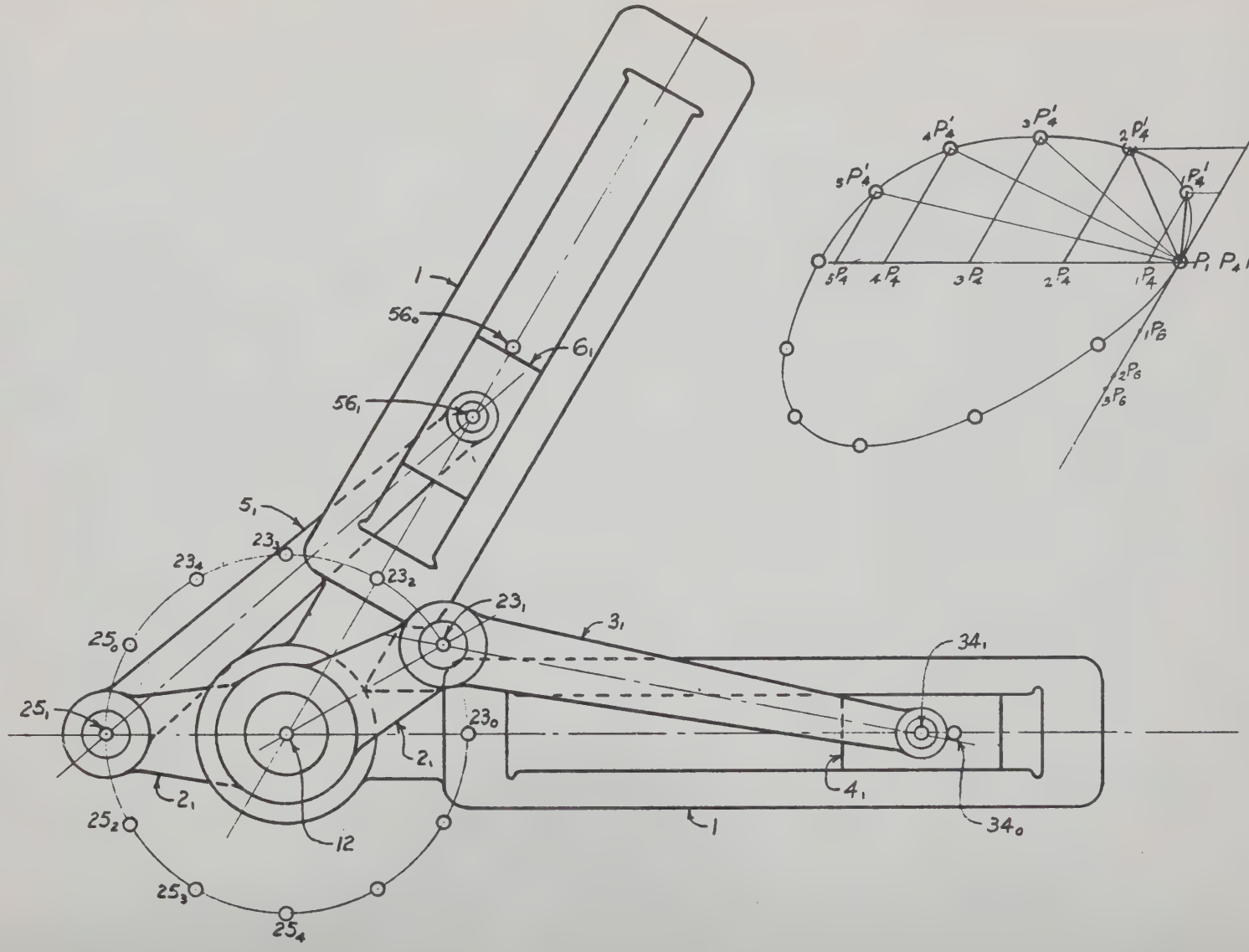

1
56₀
6₁
56₁
5₁
23₃
23₄
23₂
23₁
3₁
25₀
25₁
23₀
2₁
2₁
12
25₂
25₃
25₄
34₁
4₁
1
34₀
₄P'₄
₃P'₄
₂P'₄
₁P'₄
₅P'₄
₅P₄
₄P₄
₃P₄
₂P₄
₁P₄
P₁ P₄ P₆
₁P₆
₂P₆
₃P₆

FIG. 16

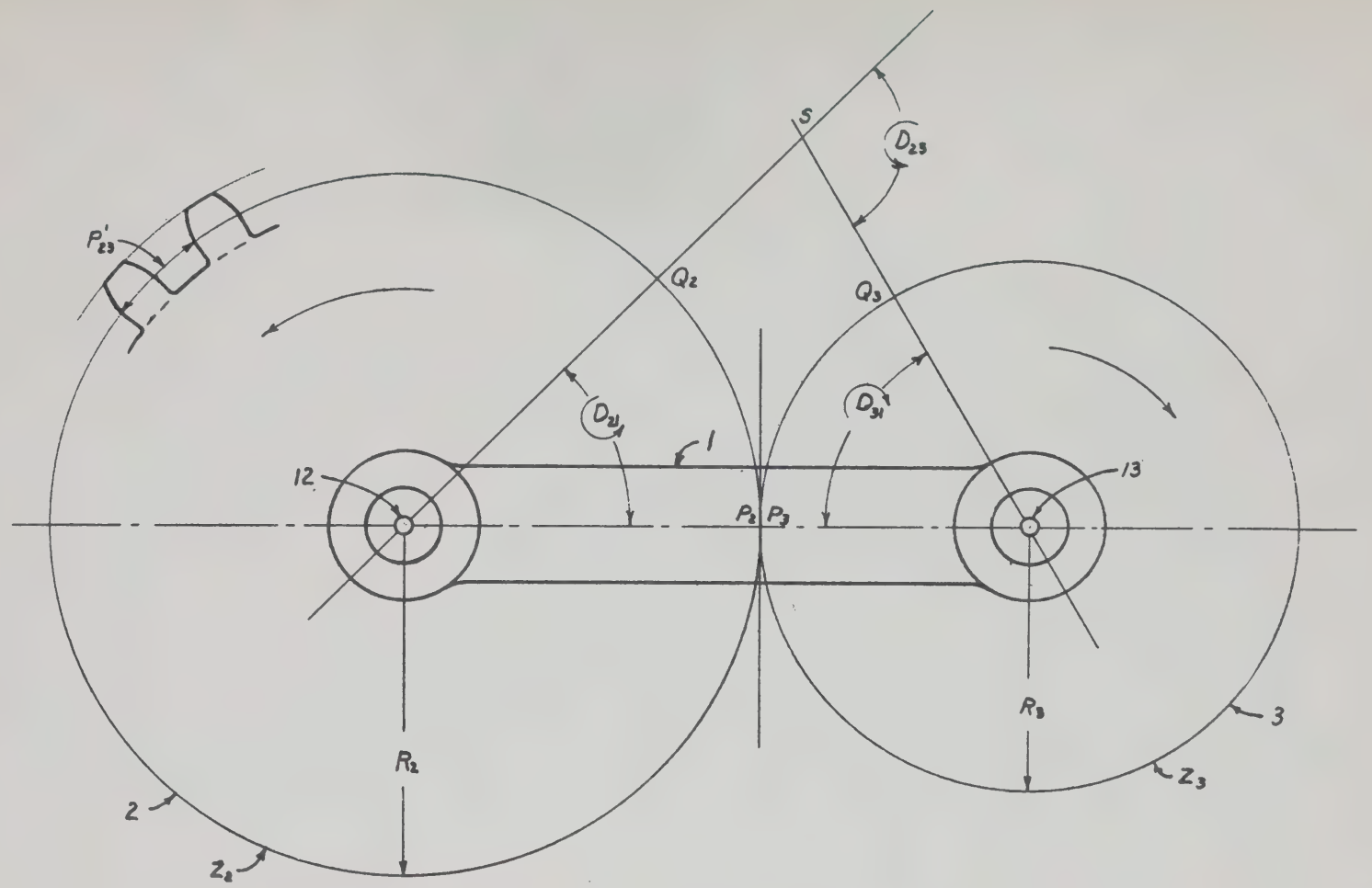


FIG. 17

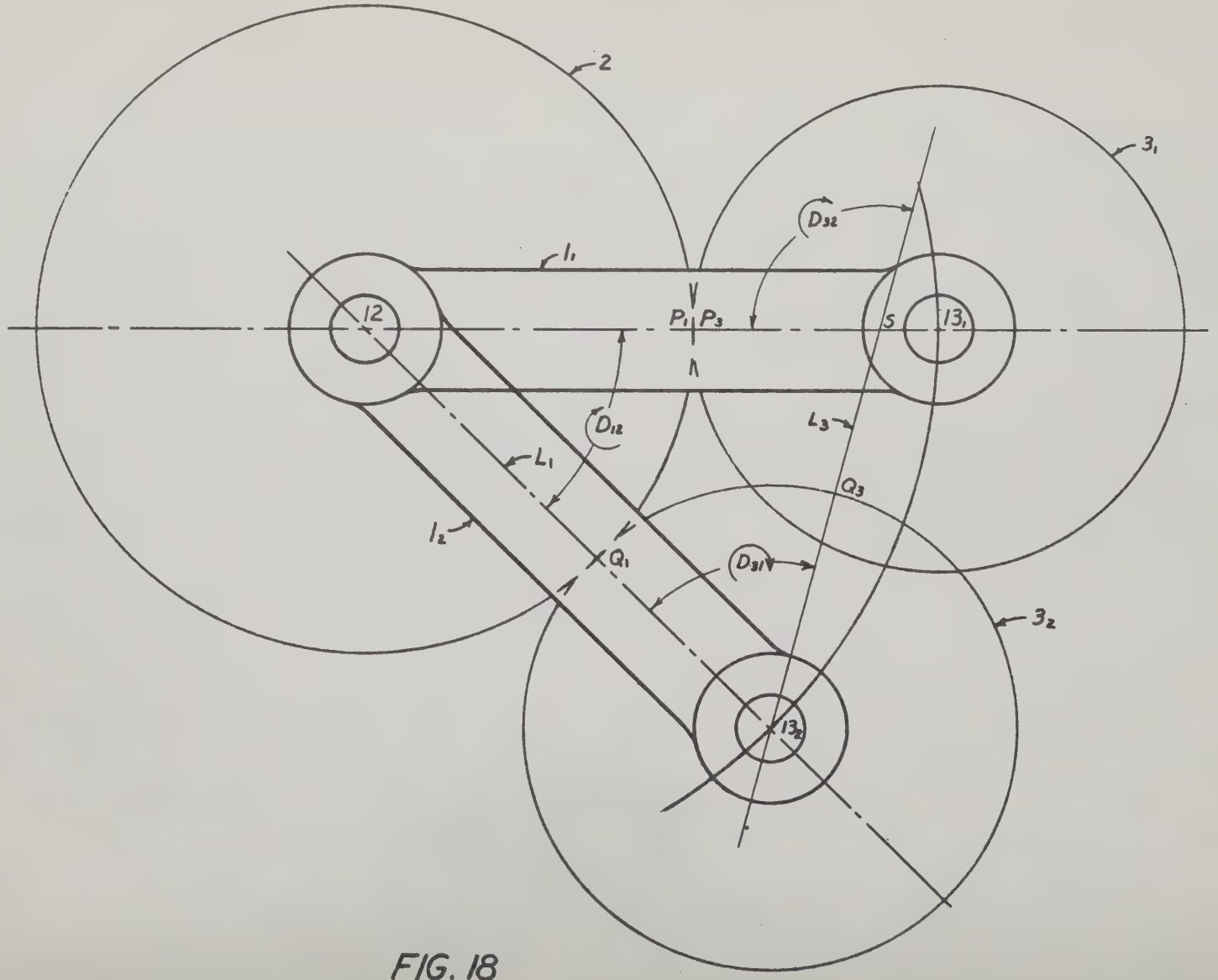


FIG. 18

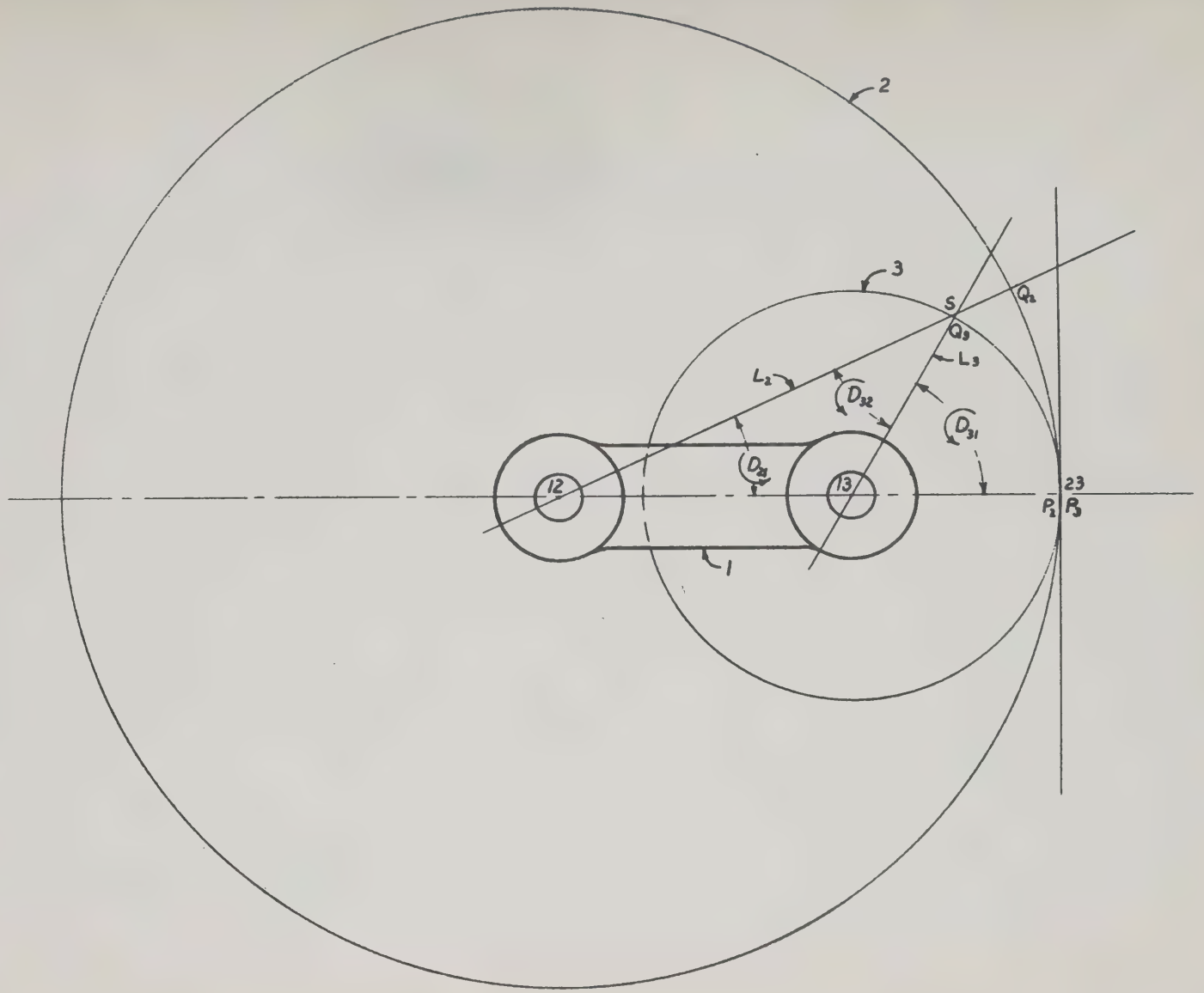


FIG. 19

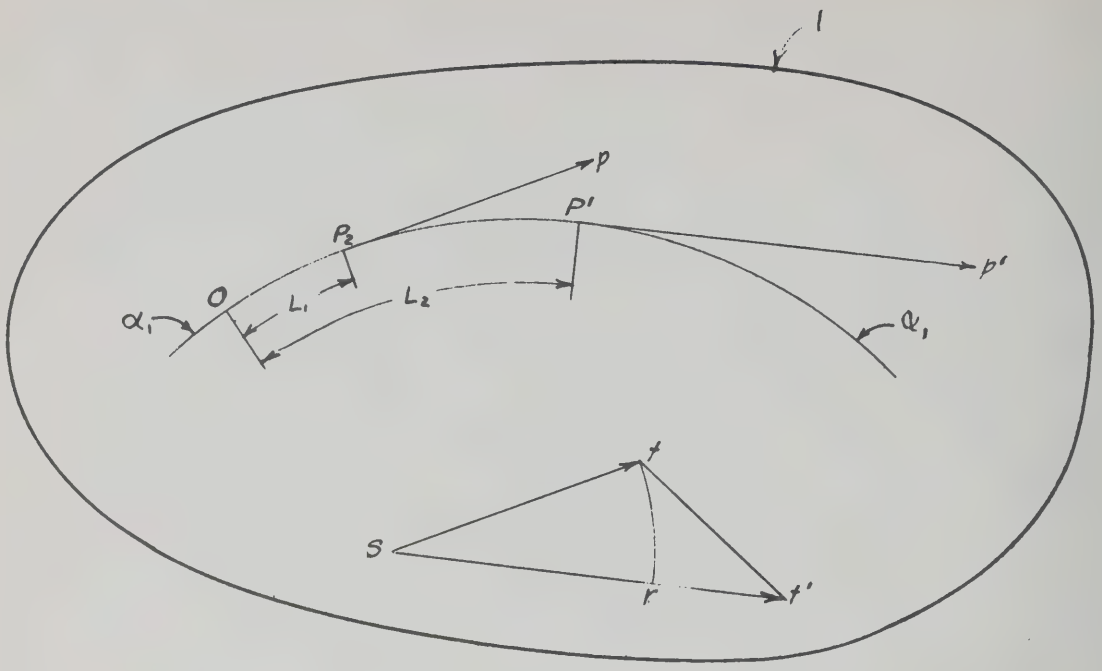


FIG. 20

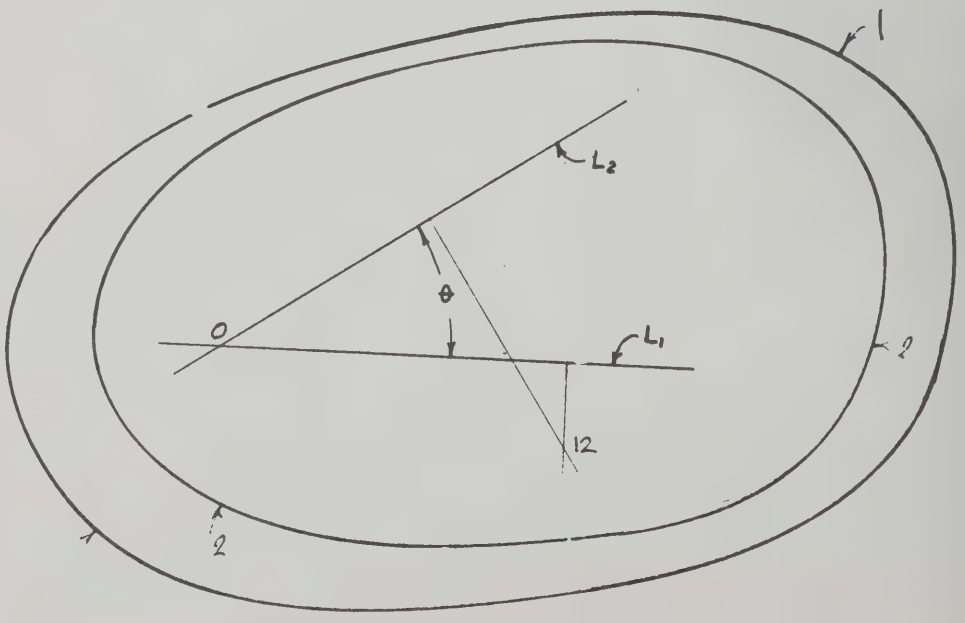


FIG. 21

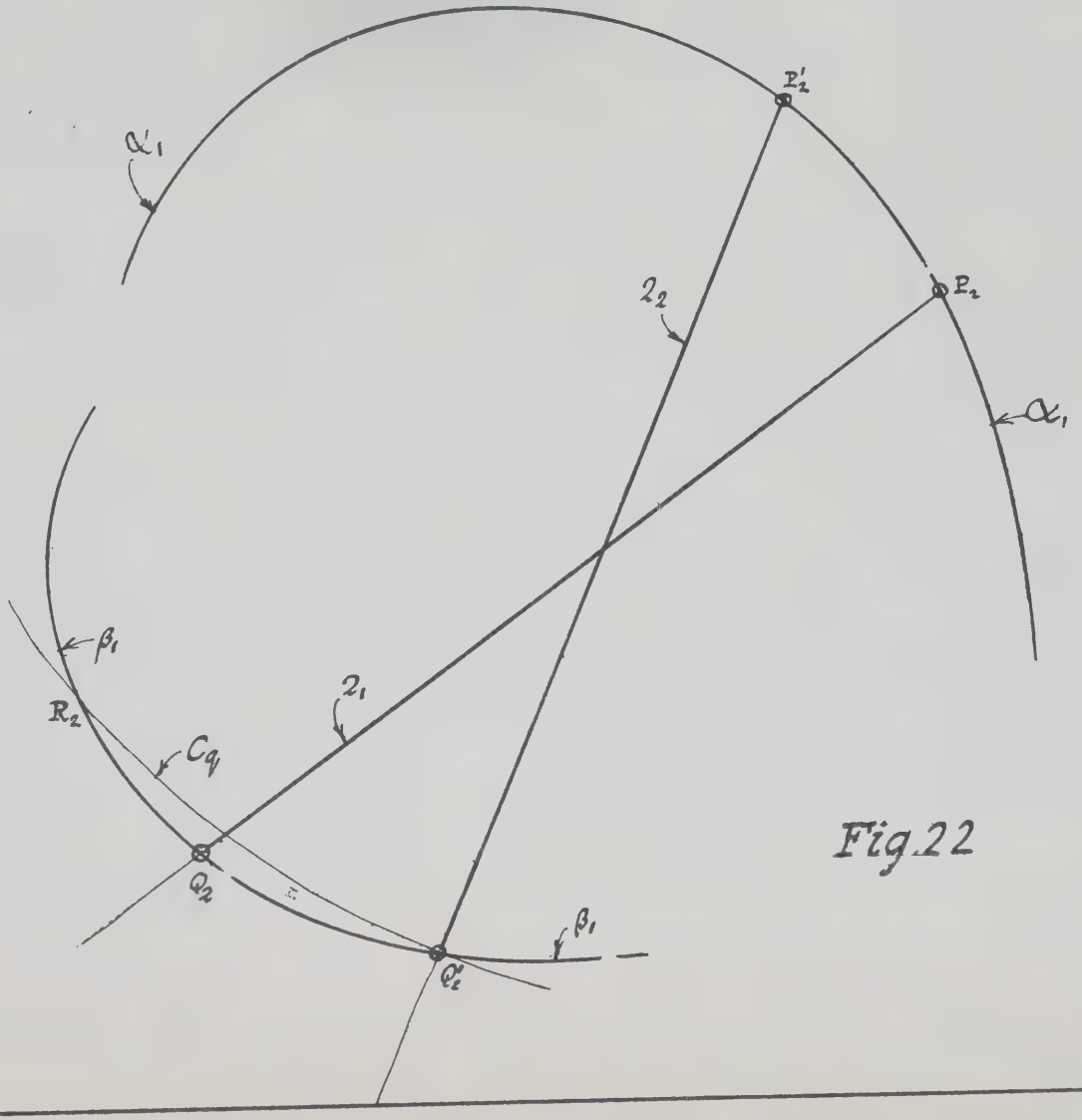


Fig.22

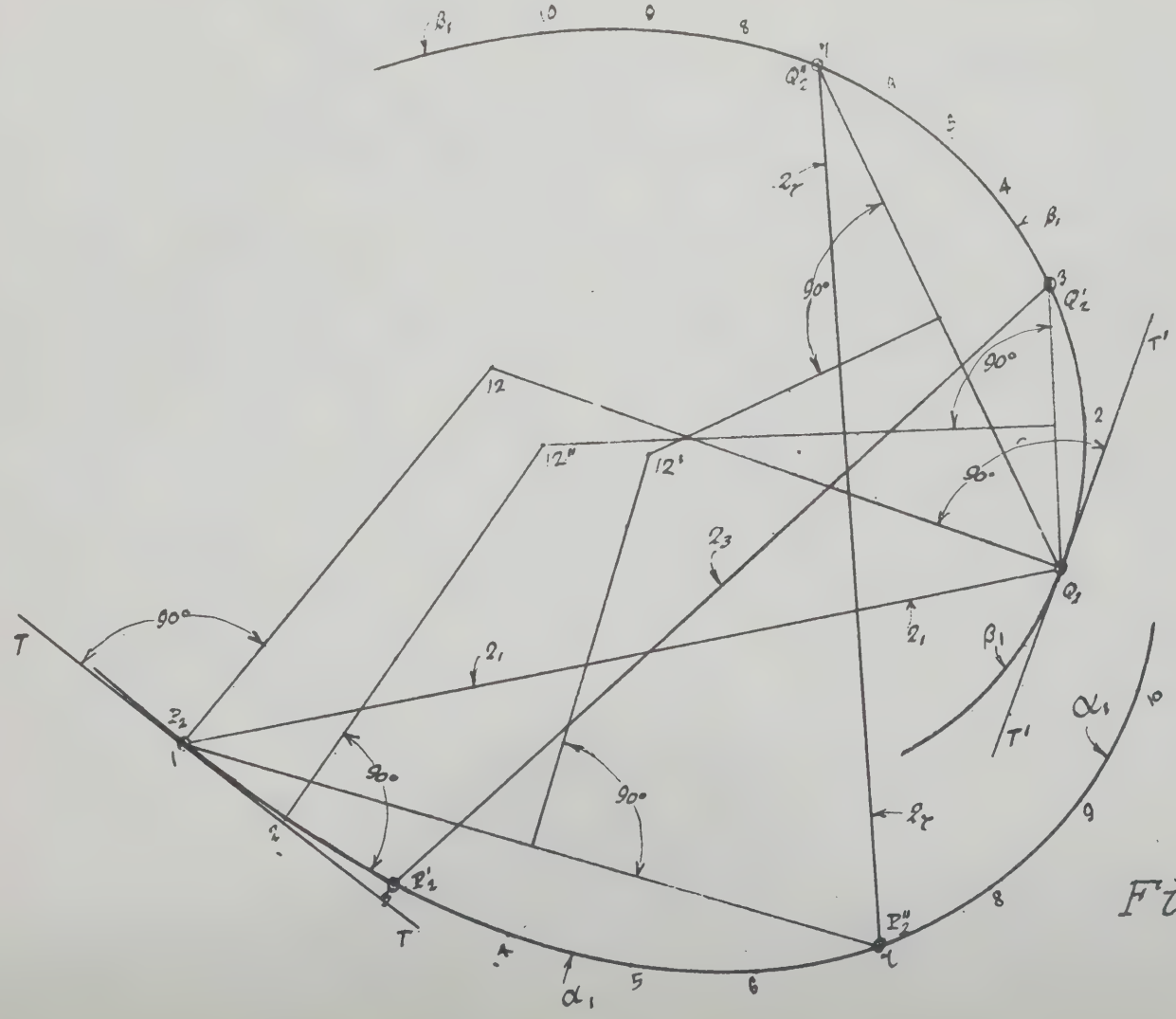

β1
10
9
8
Q'2
Z2
90°
Q'2
T'
12
12''
12'
Z3
Q3
Z1
β1
T
90°
Z1
Z2
1
90°
90°
Z2
P'2
T
P''2
α1
5
6
α1
10
9
8

Fig. 23.

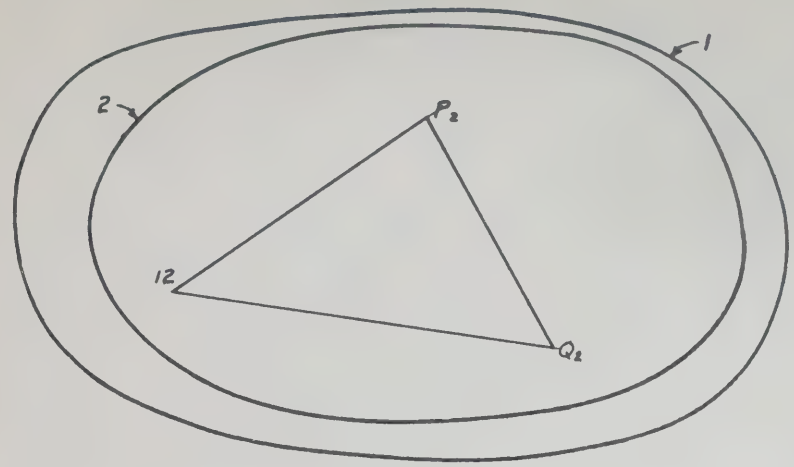


FIG. 24

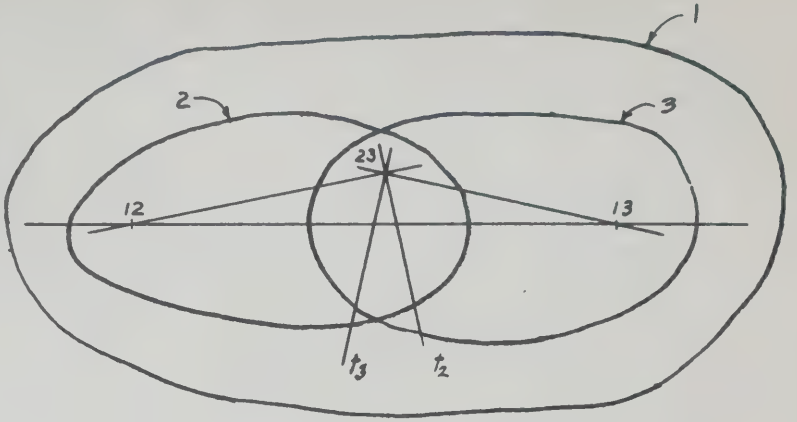


FIG. 25a

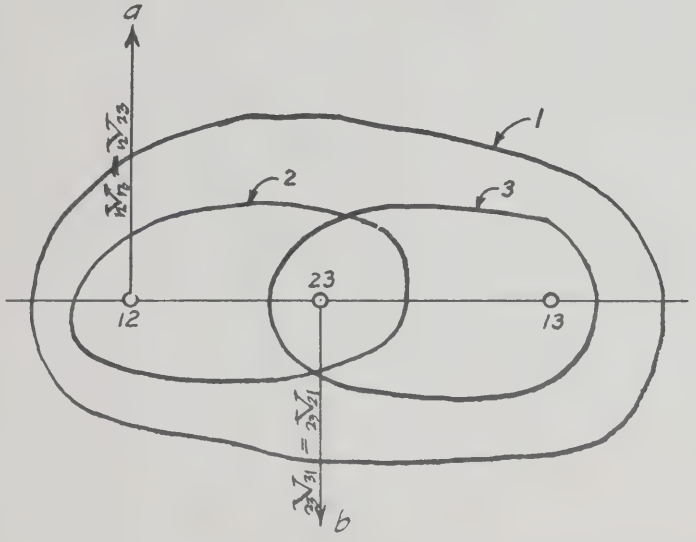


FIG. 25 b

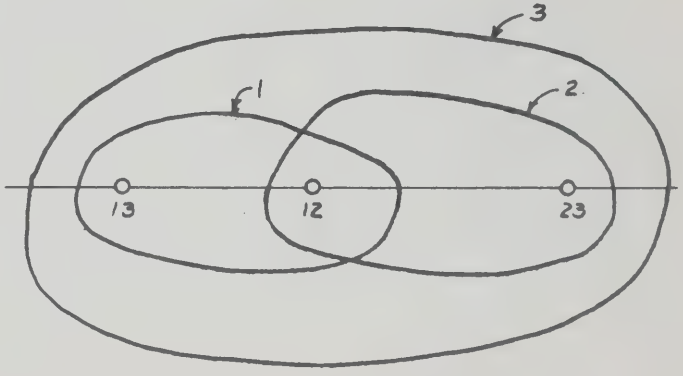
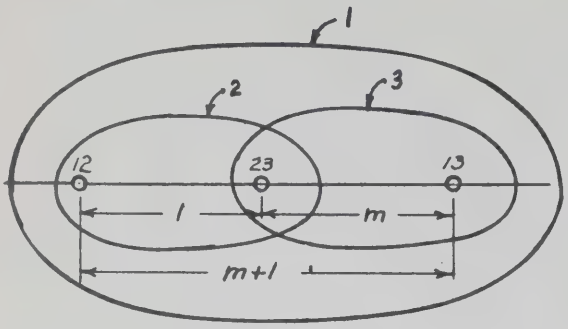
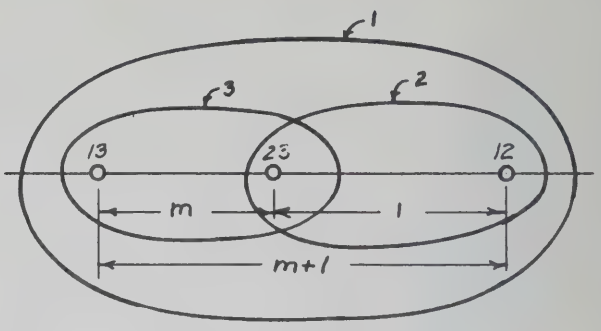


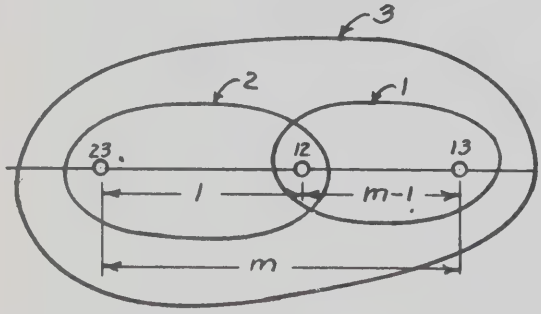
FIG. 25 c



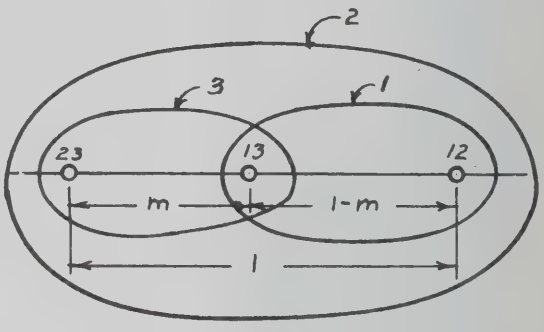
a



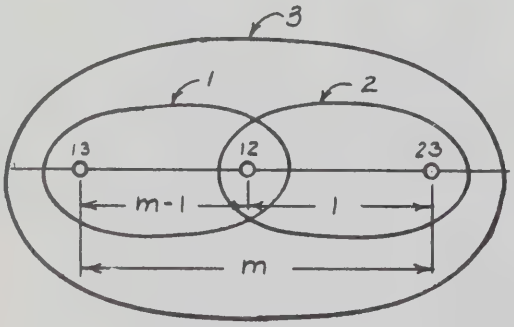
b



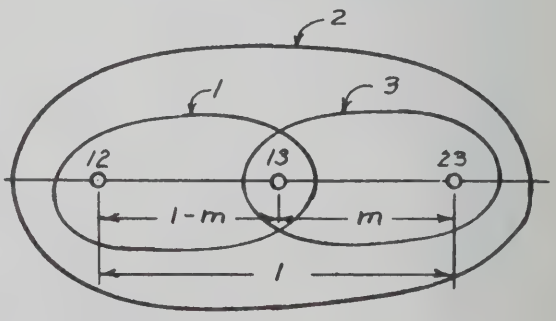
c



d



e



f

FIG. 26

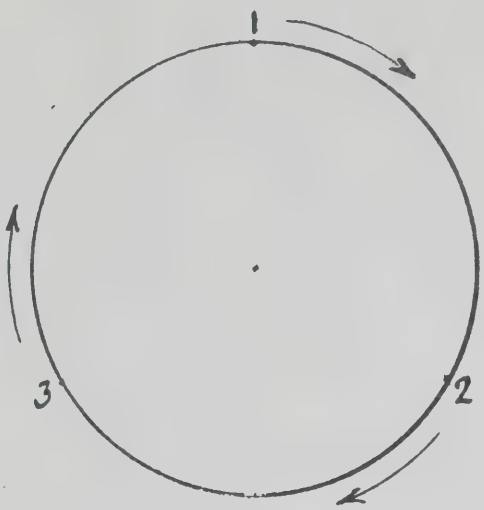


Fig. 27

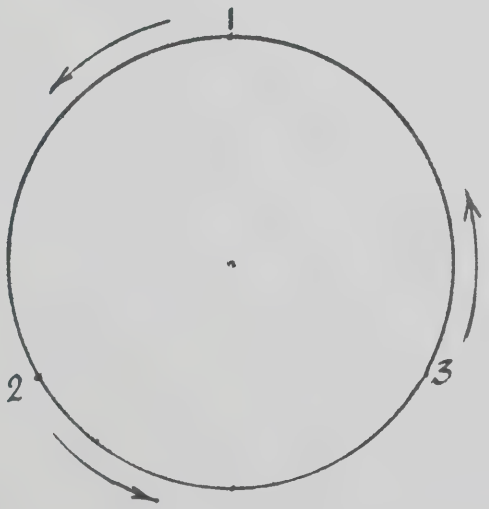


Fig. 27

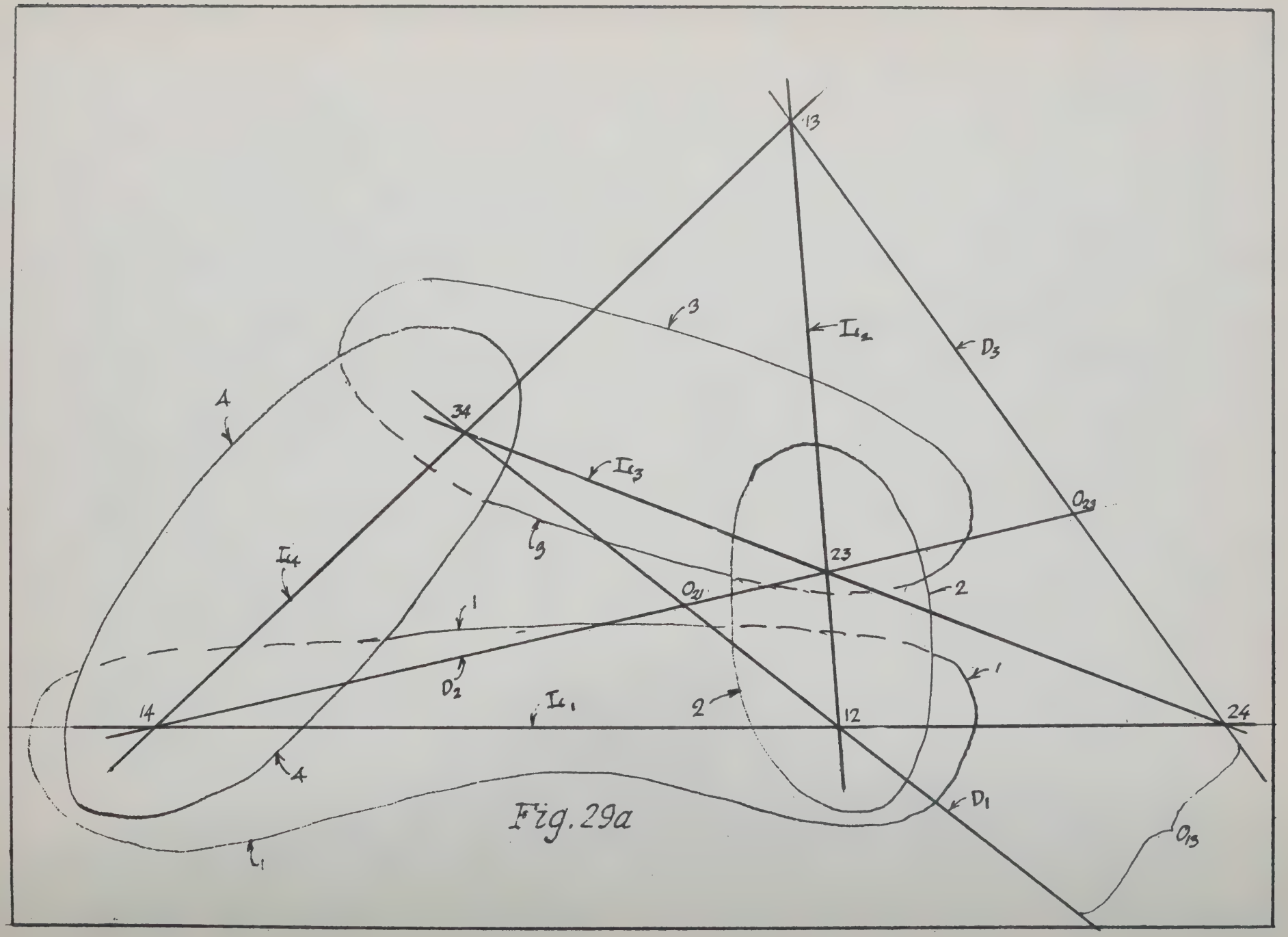


Fig. 29a

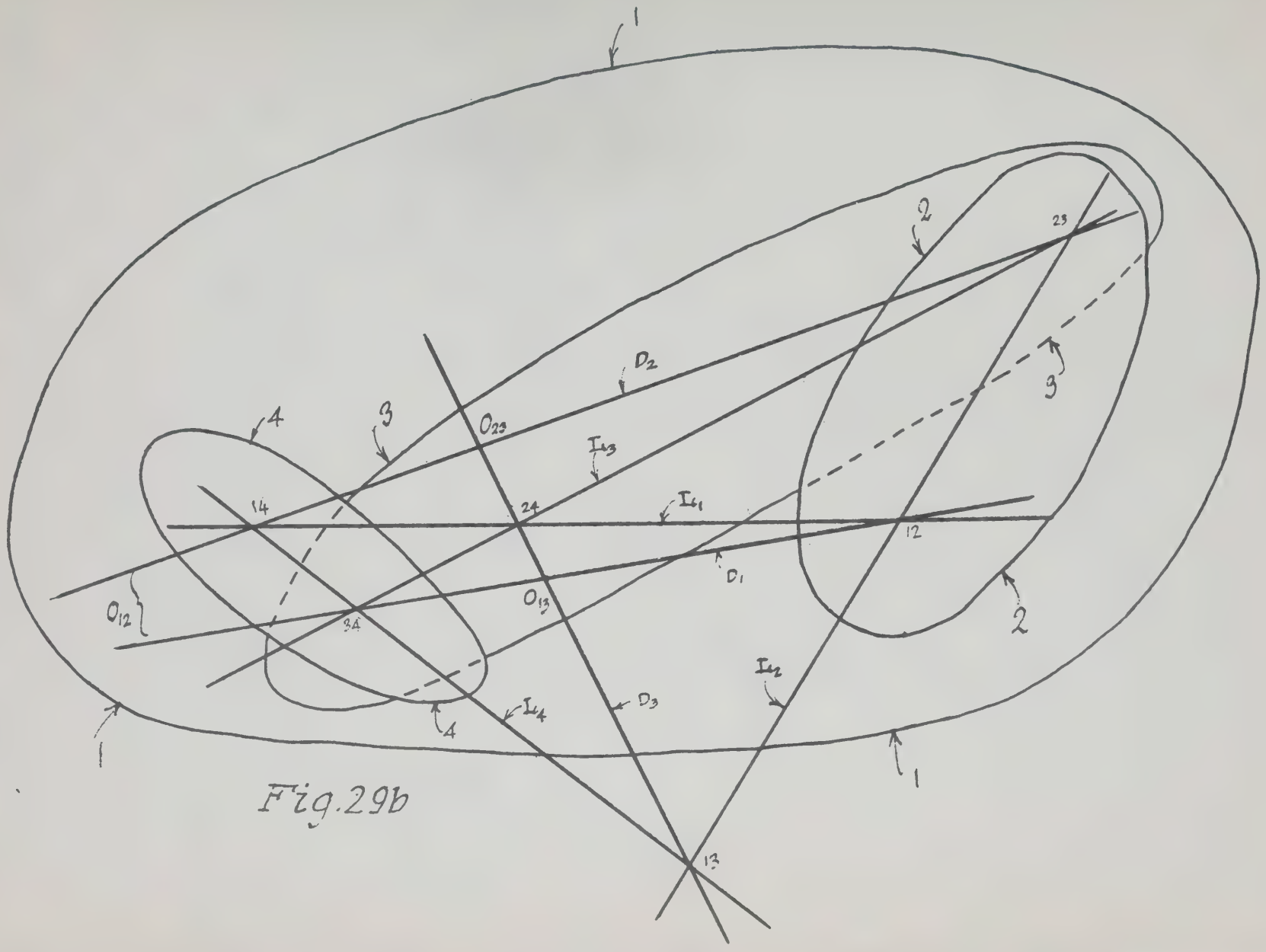


Fig.29b

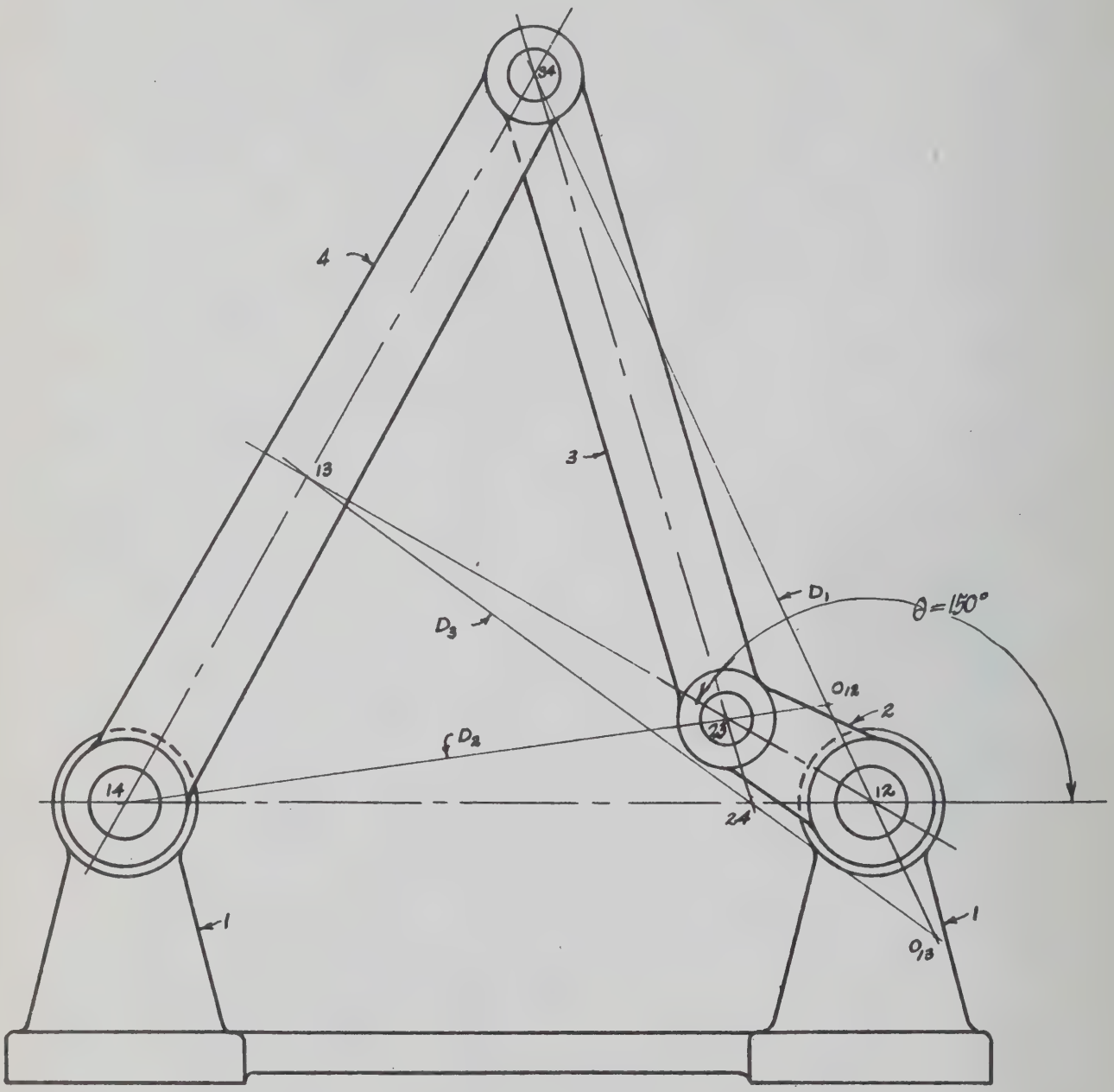


FIG. 29c

Scale: 3" = 1'-0"

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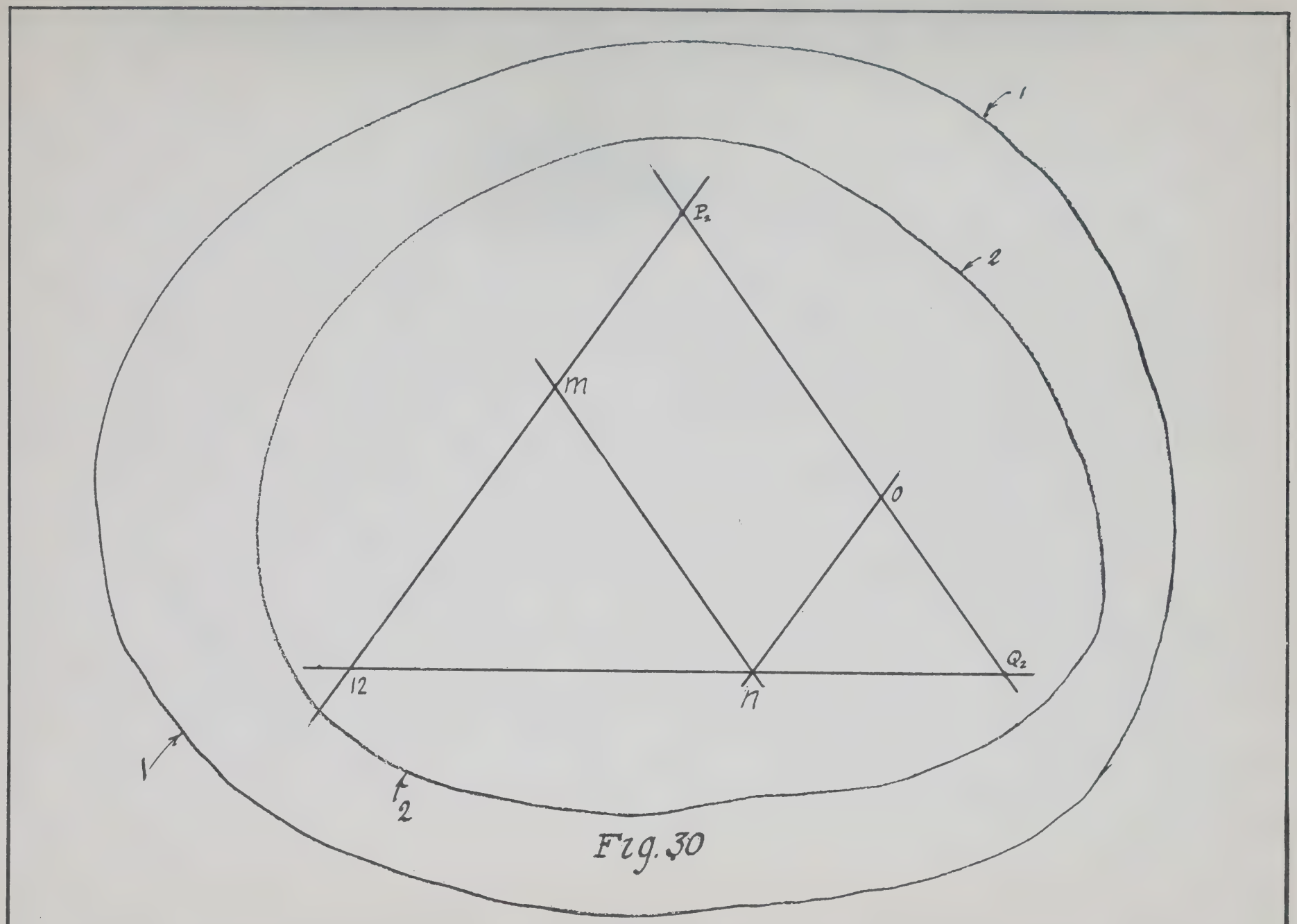


Fig. 30

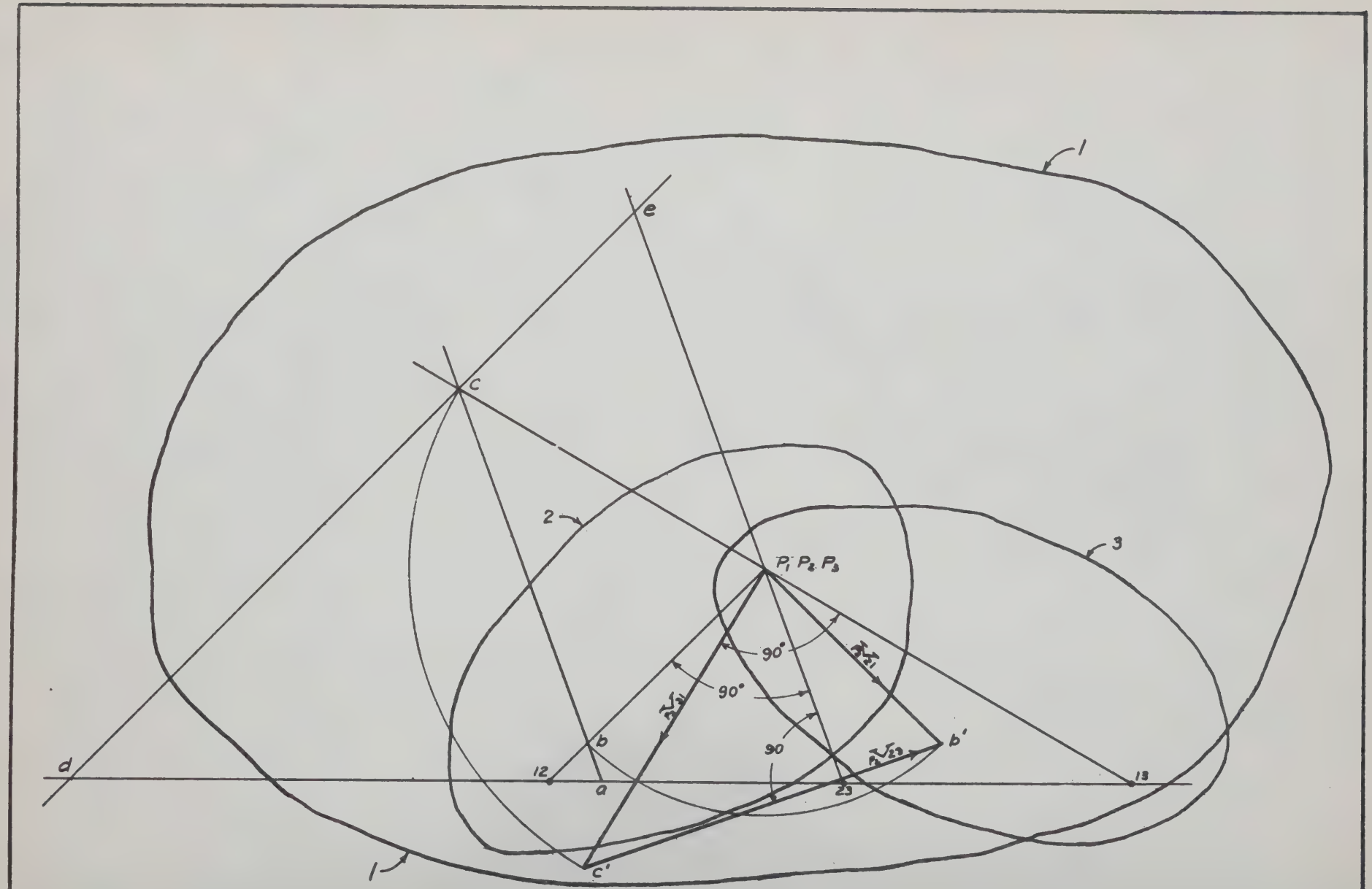

1
e
c
2
3
$P_1 P_2 P_3$
90°
90°
90
$\sqrt{2}\vec{v}_{21}$
$\sqrt{2}\vec{v}_{21}$
$\sqrt{2}\vec{v}_{23}$
b
b'
d
12
a
23
13
c'
1

FIG. 31

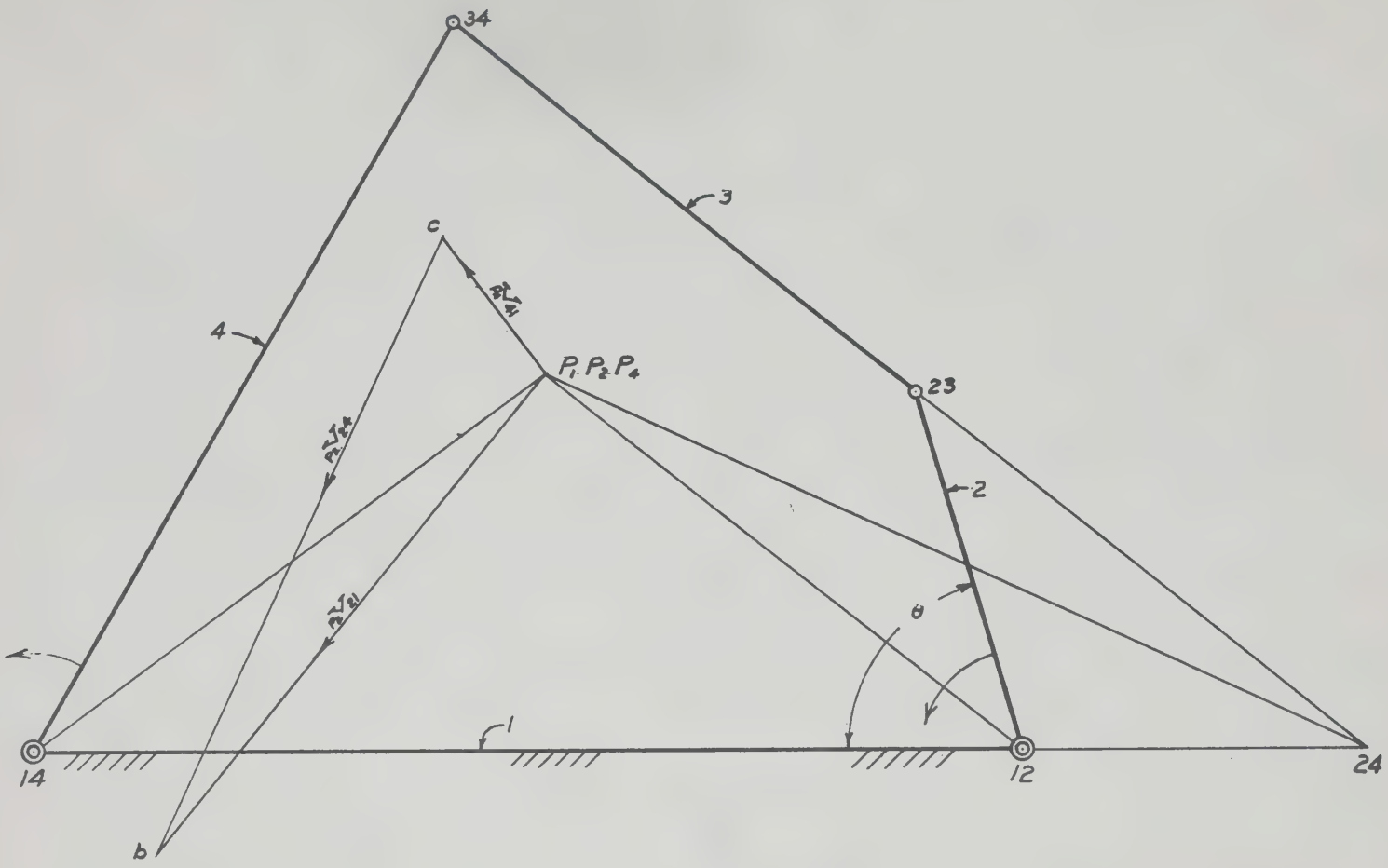


FIG. 32

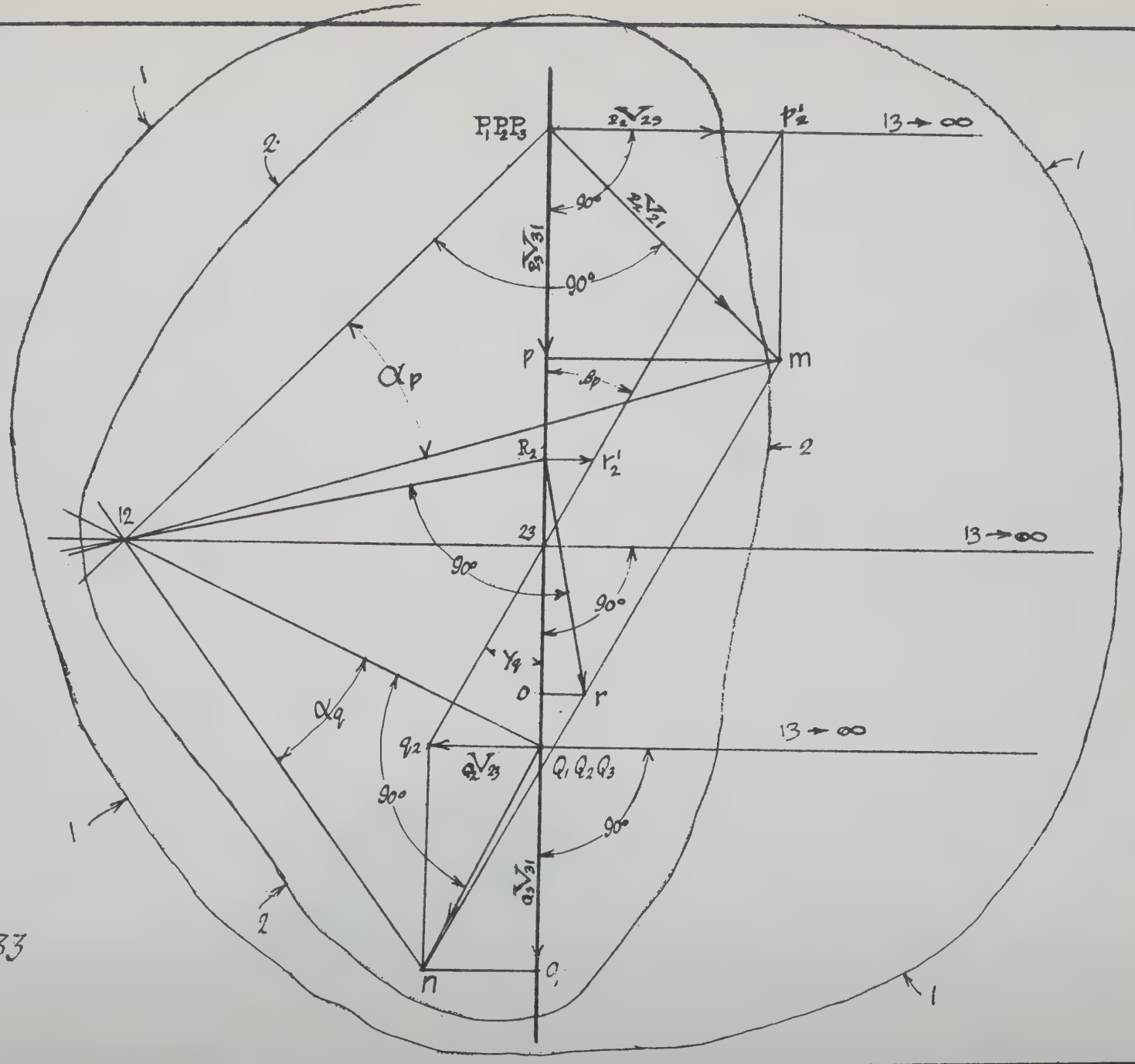

1
2
P1P2P3
z2V23
P'2
13 → ∞
90°
z2V21
z3V31
90°
αp
p
m
βp
R2
r'2
12
23
13 → ∞
90°
90°
γq
0
r
αq
q2
q2V23
Q1Q2Q3
13 → ∞
90°
90°
q3V31
n
Q

Fig. 33

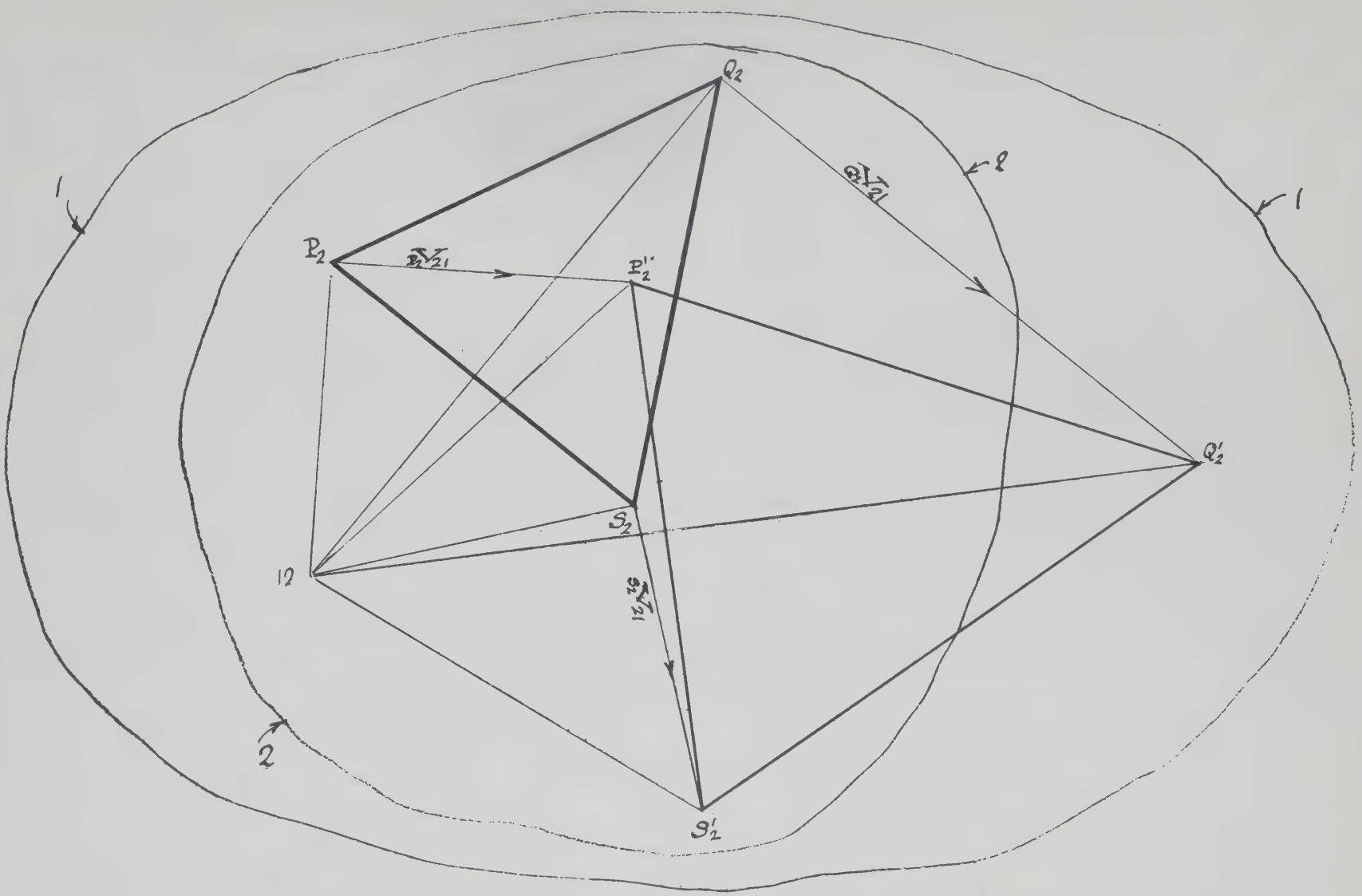


Fig. 33a

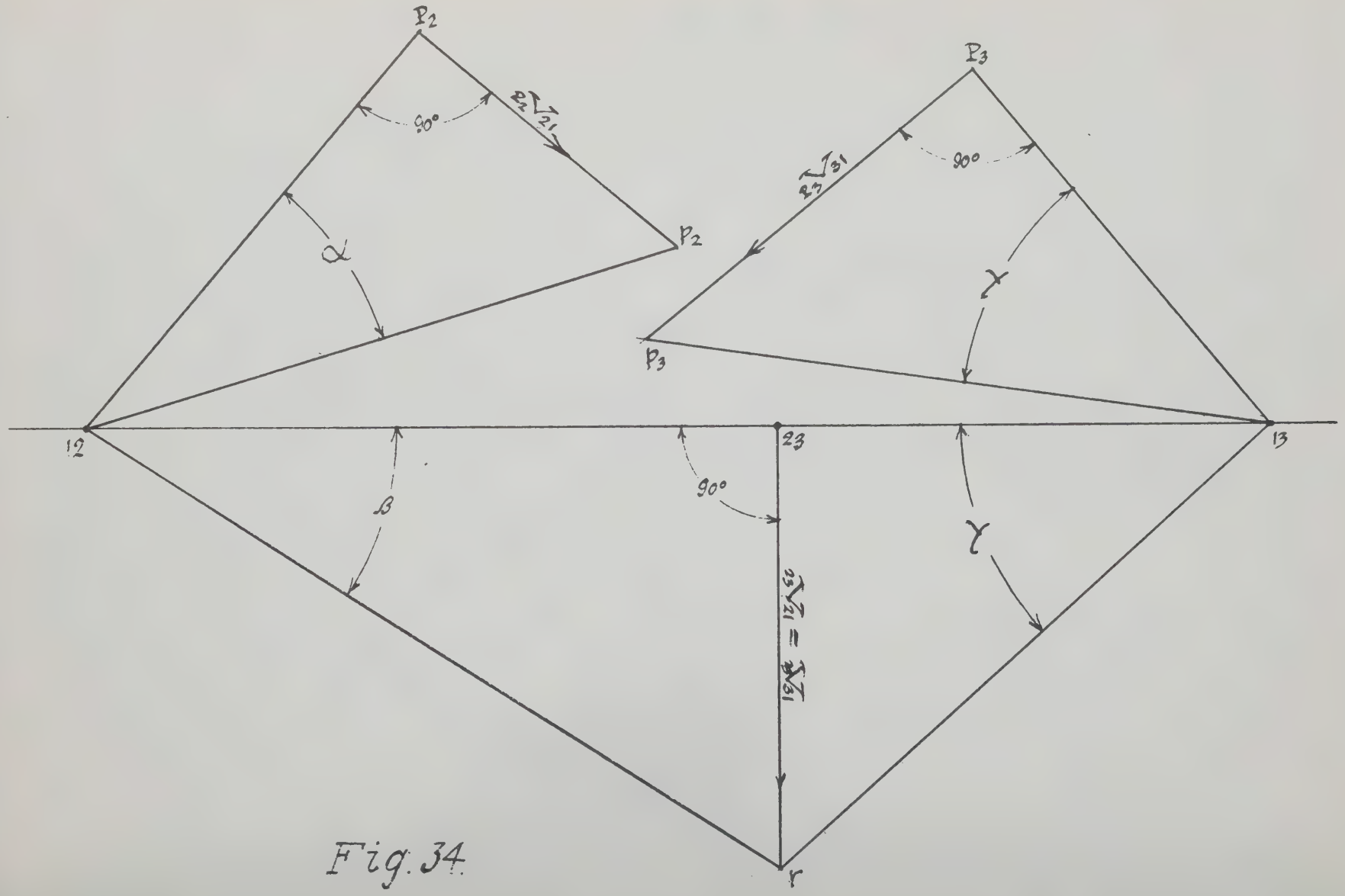
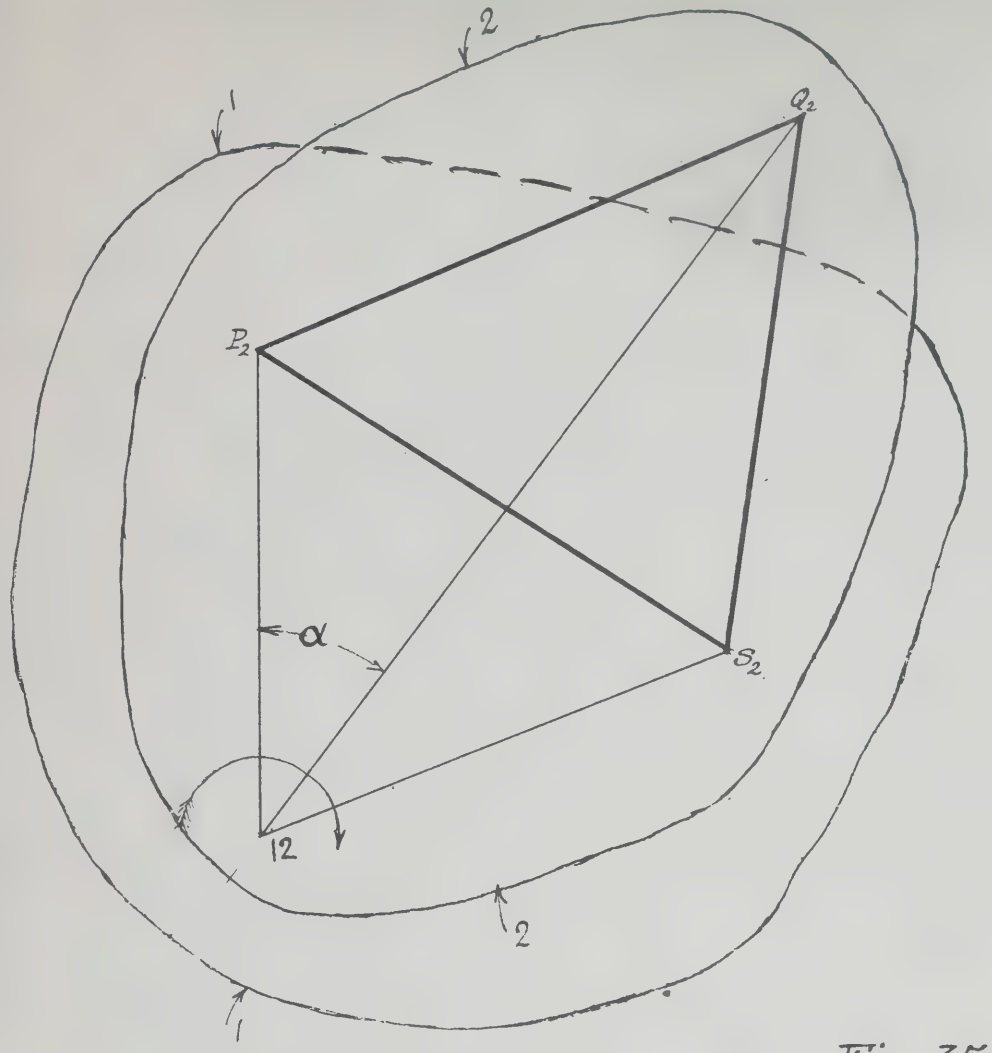
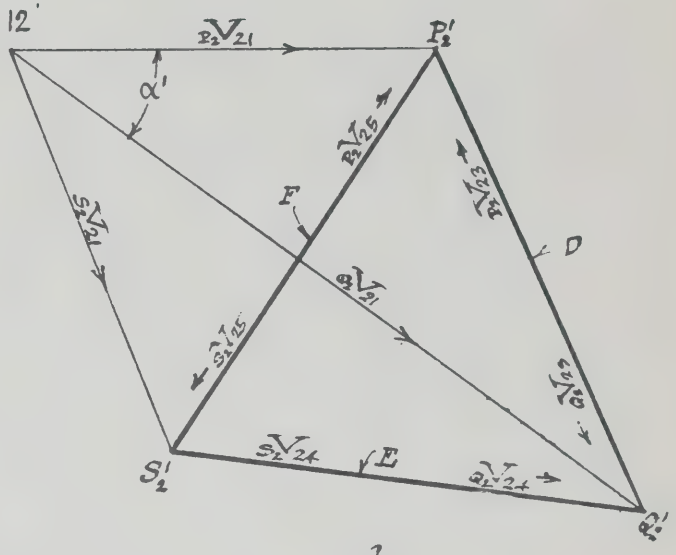

P2
P2
P2
ε2 v21
90°
α
P3
P3
P3
ε3 v31
90°
γ
12
23
13
β
90°
γ
ε23 v23 = ε31 v31
r

Fig. 34


2
1
Q_2
P_2
α
S_2
12
2
1

a


12'
P_2V_{21}
P_2'
α'
P_2V_{25}
P_2V_{23}
F
D
S_2V_{21}
a_2V_{21}
S_2V_{25}
a_2V_{23}
S_2V_{24}
E
a_2V_{24}
S_2'
Q_2'

b

Fig 35

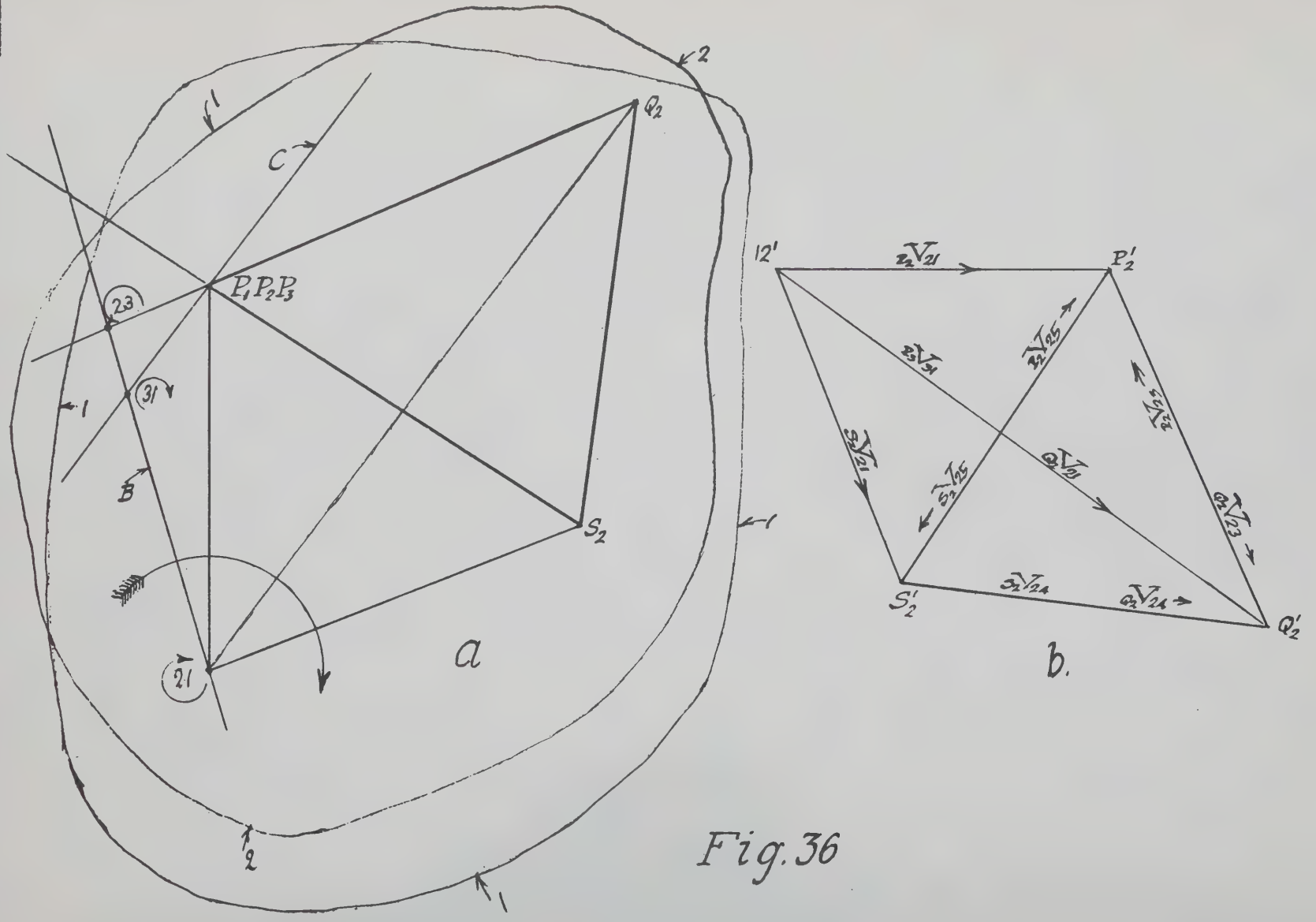

2
1
C
Q_2
$P_1P_2P_3$
23
31
1
B
S_2
21
a
1
2
1
12'
$_2V_{21}$
P_2'
$_3V_{31}$
$_2V_{25}$
$_2V_{23}$
$_2V_{21}$
$_2V_{25}$
Q_2V_{21}
Q_2V_{23}
$_2V_{24}$
Q_2V_{24}
S_2'
Q_2'
b.

Fig. 36

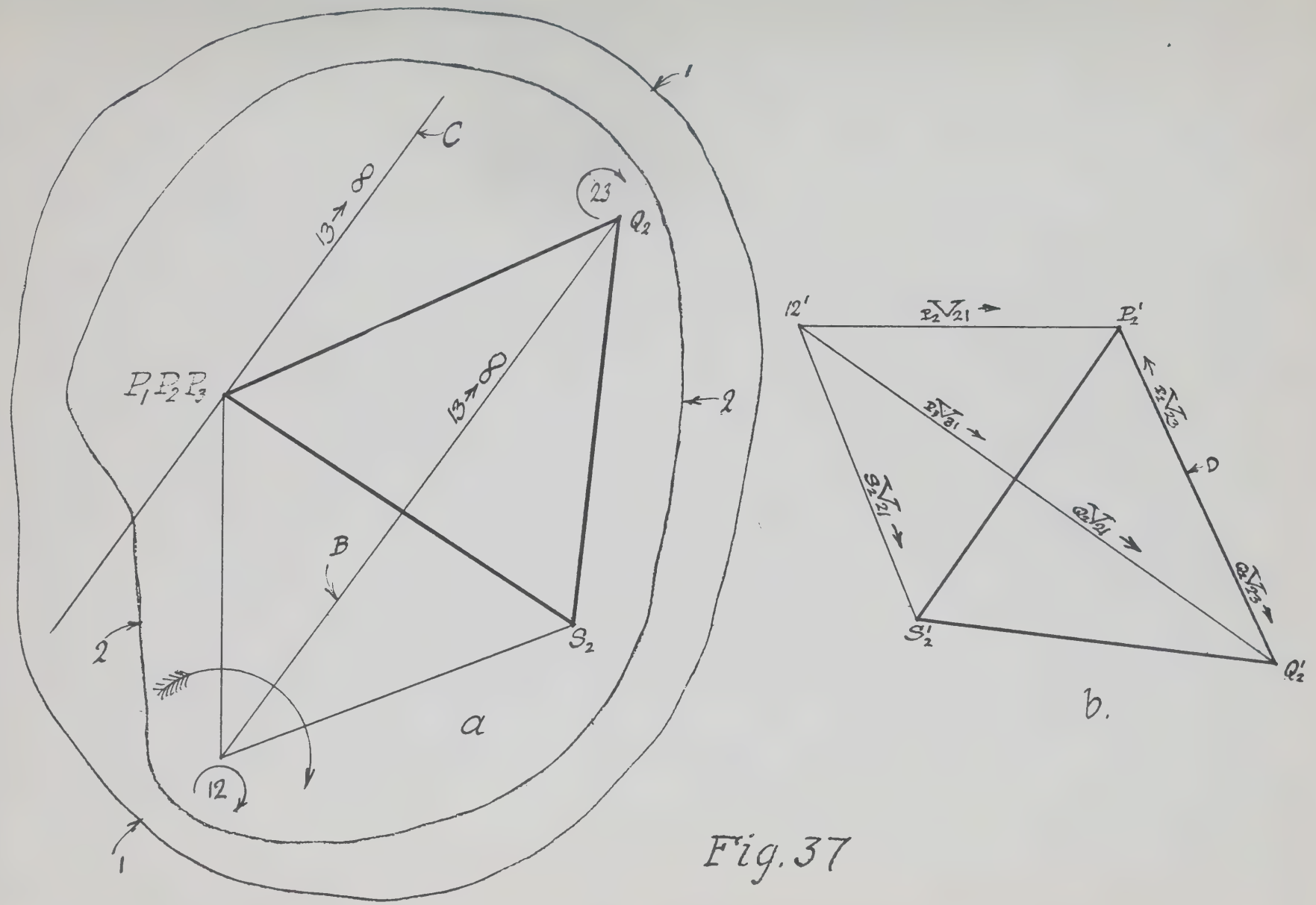

1
C
13→∞
23
Q_2
$P_1P_2P_3$
13→∞
2
B
2
S_2
a
12
1
12'
P_2V_{21}
P_2'
P_2V_{23}
P_2V_{31}
D
Q_2V_{21}
Q_2V_{21}
Q_2V_{23}
S_2'
Q_2'
b.

Fig. 37

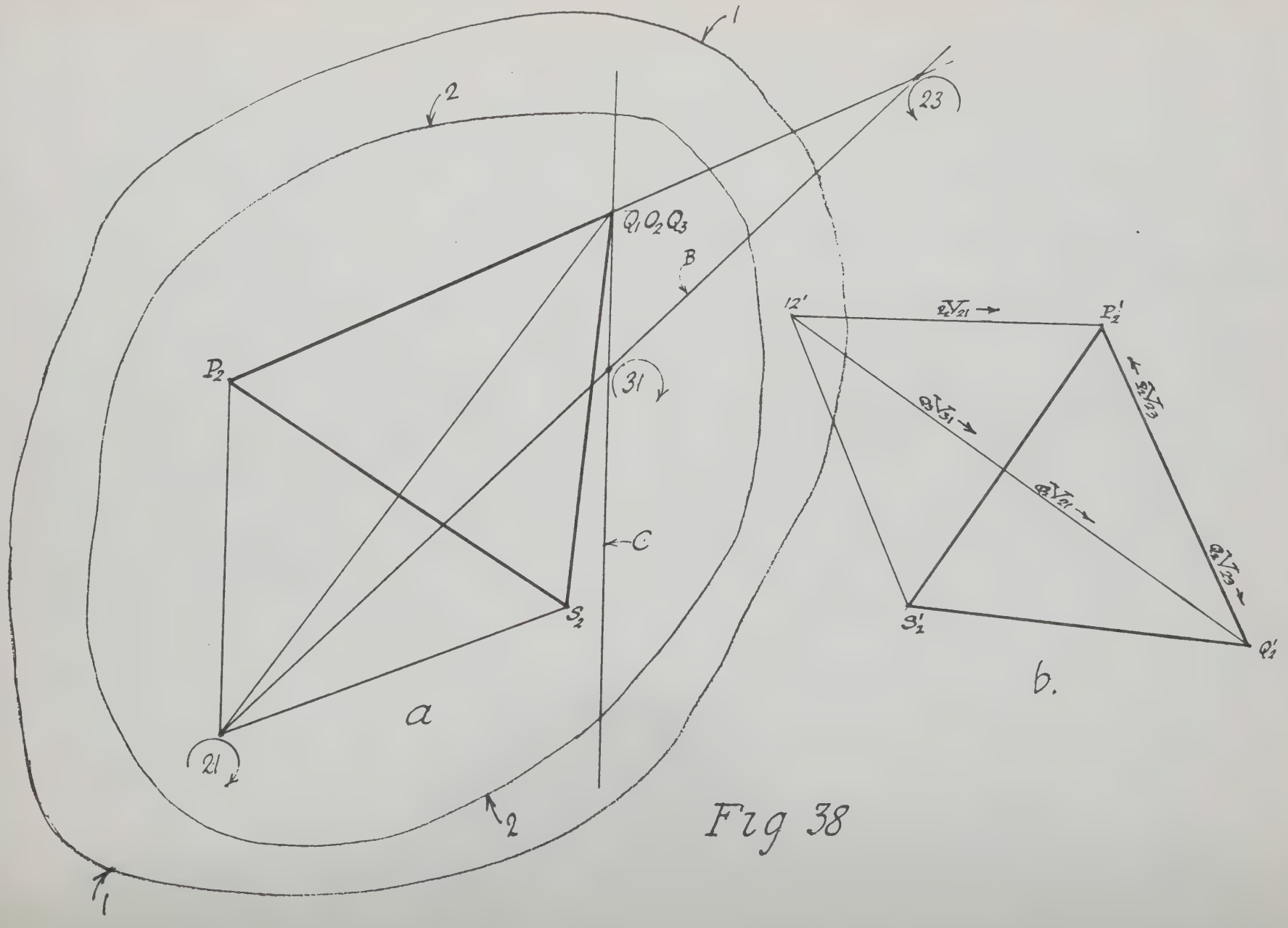

1
2
23
$Q_1 Q_2 Q_3$
B
P_2
31
C
S_2
a
21
2
1
12'
$q_2 v_{21}$
P_2'
$q_3 v_{31}$
$q_3 v_{23}$
$q_2 v_{21}$
$q_2 v_{23}$
S_2'
Q_2'
b.

Fig 38

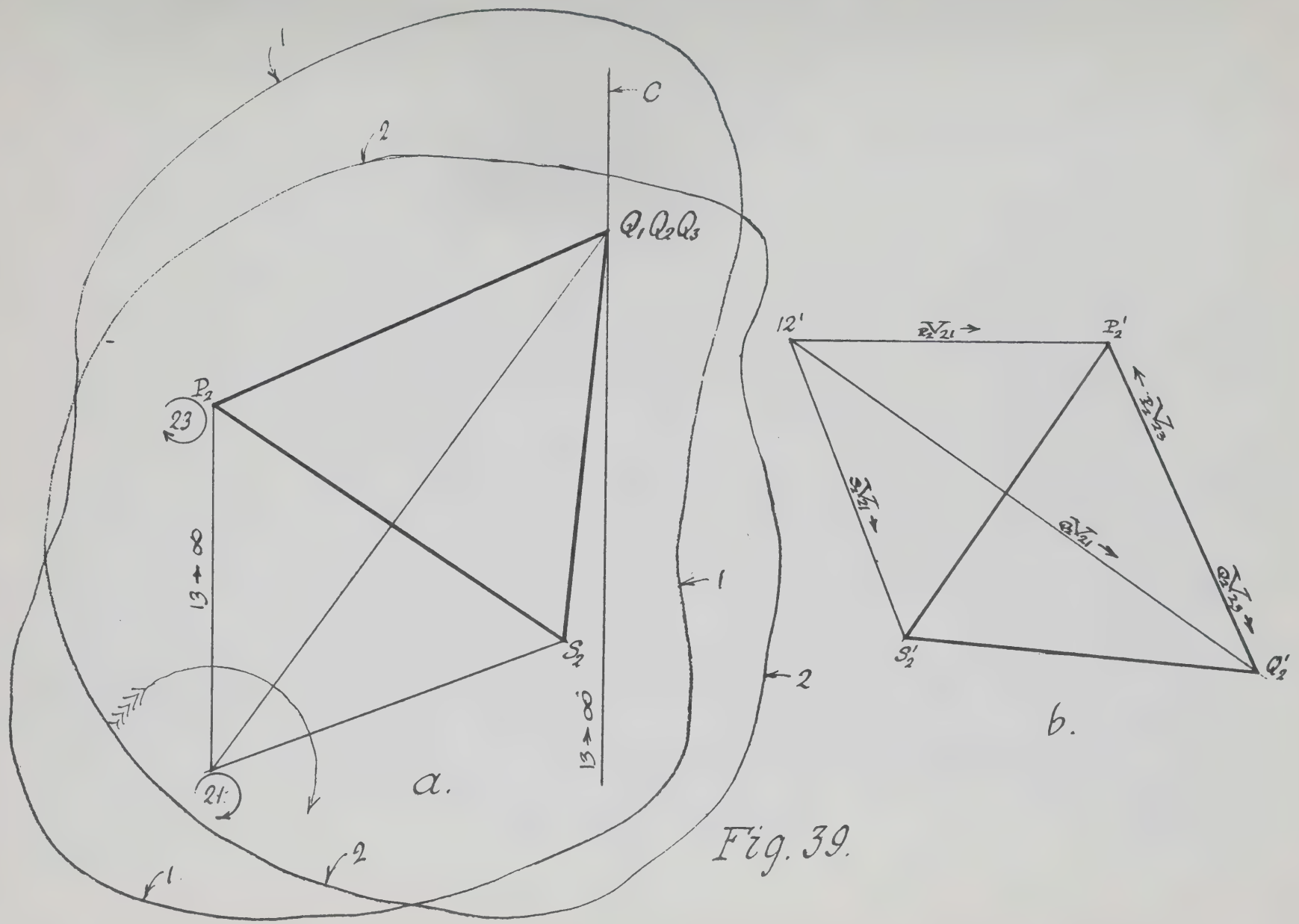


Fig. 39.

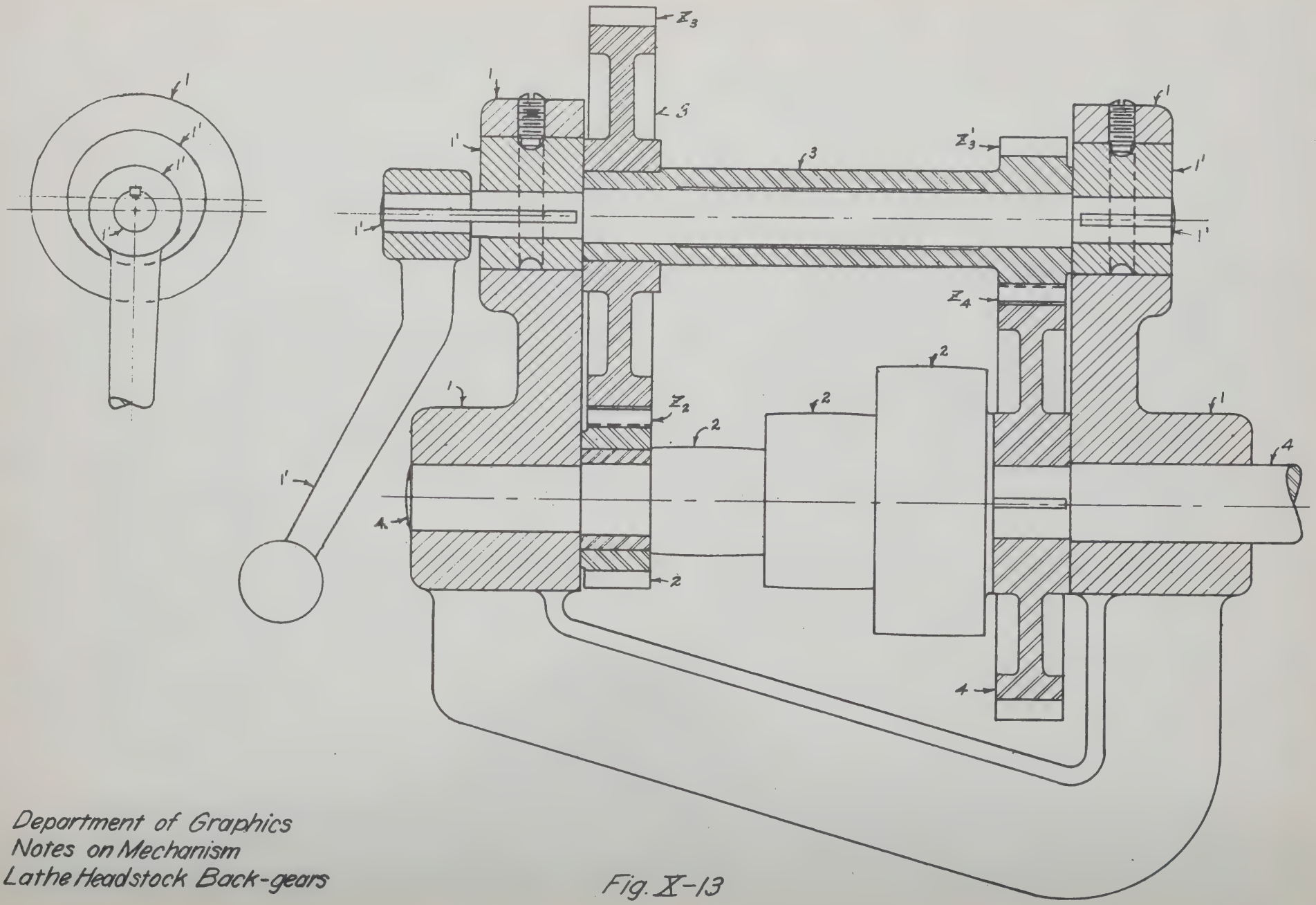


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Fig. X-13

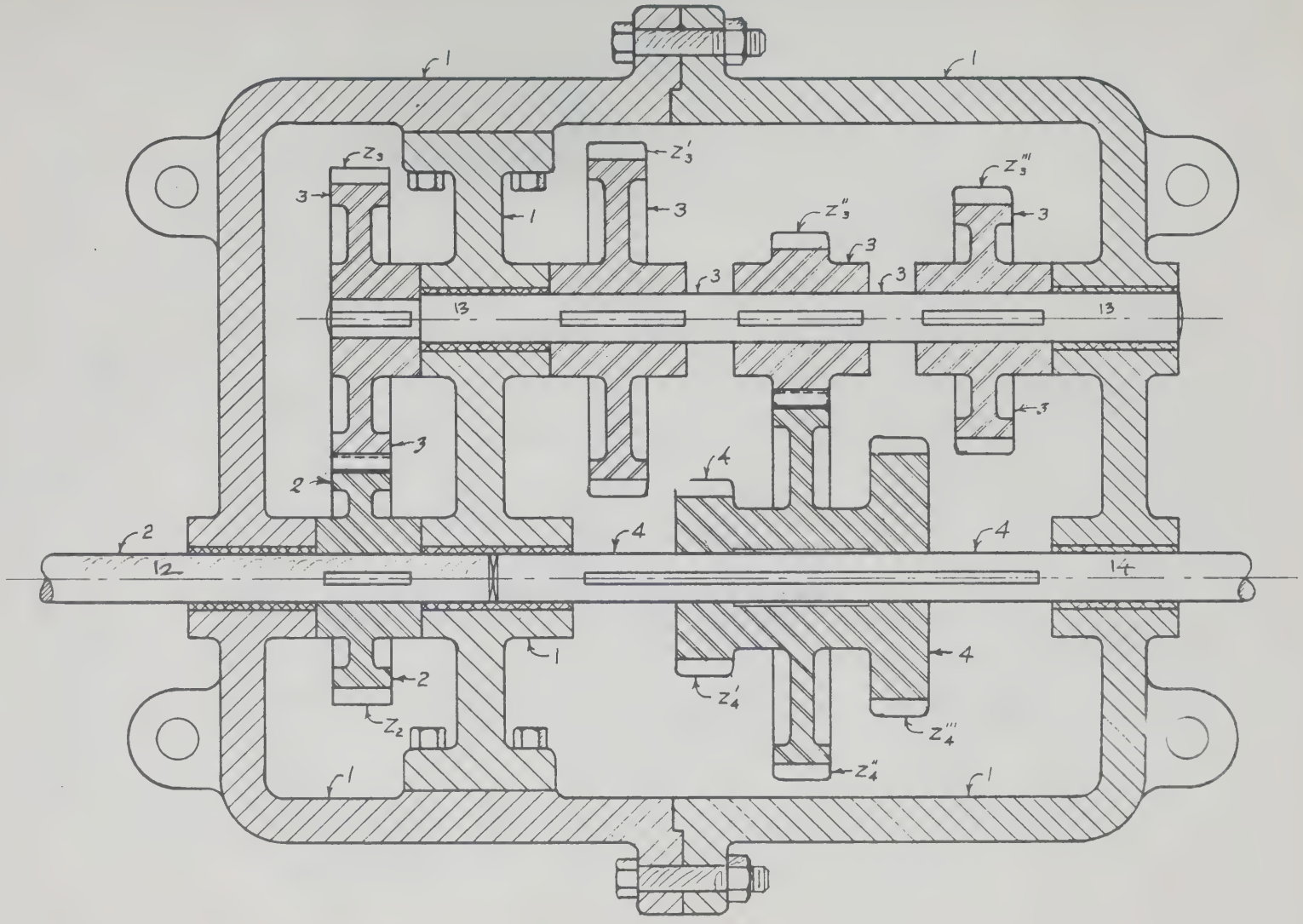

FIG. X-15

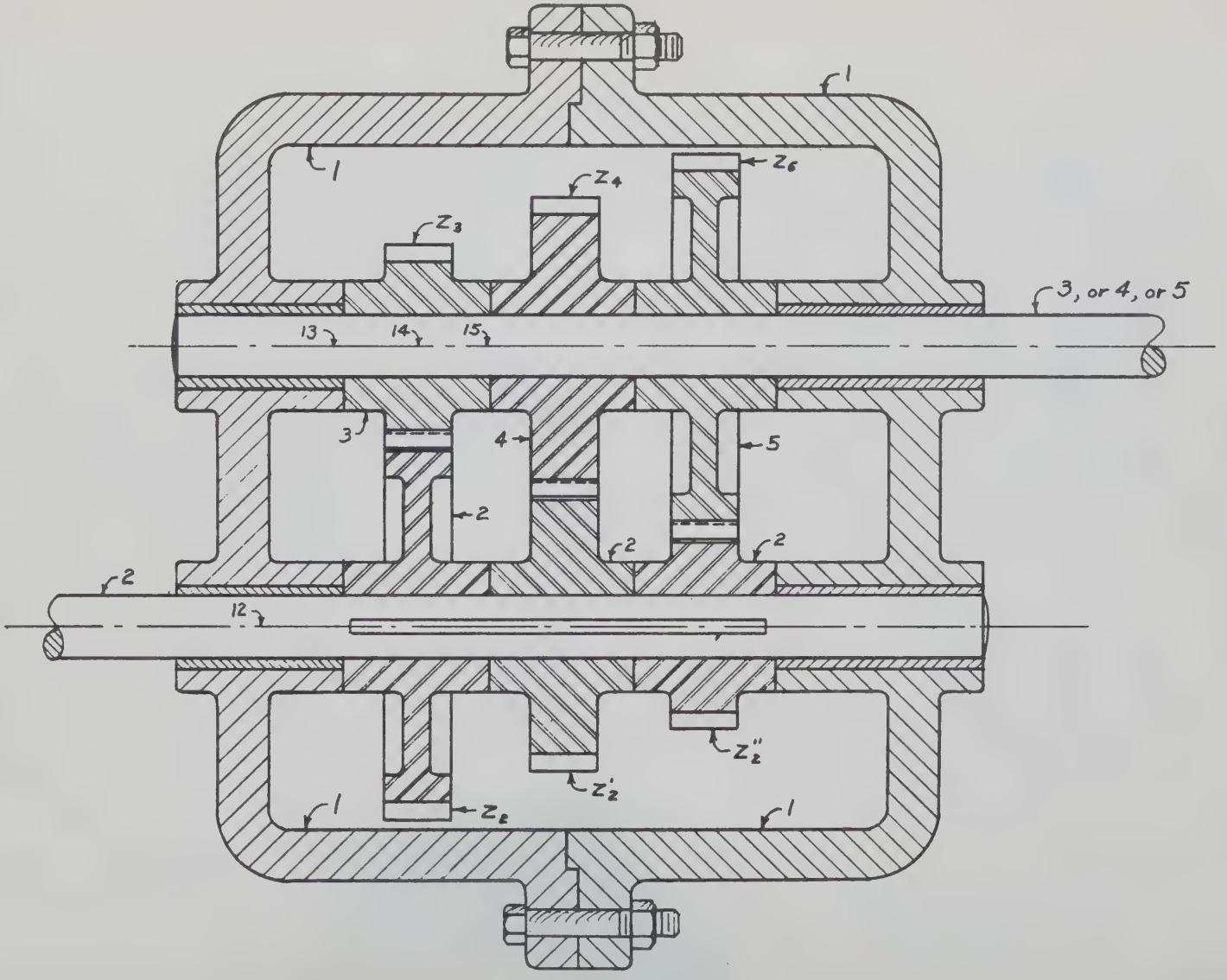

FIG. X-16

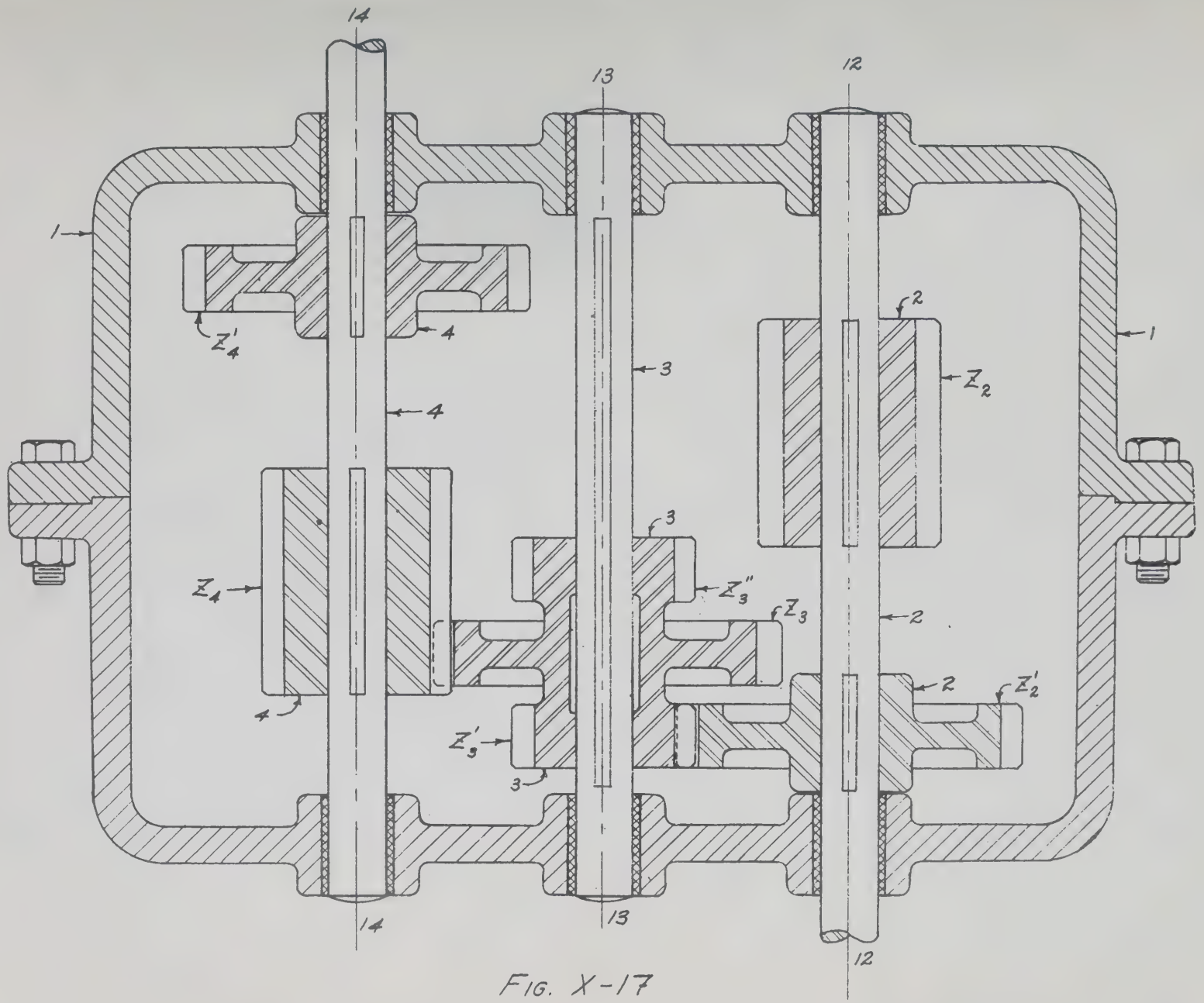

FIG. X-17

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Notes on Mechanism

Bibasal - Septemic Gear Train

Baselinks: 1 and 2

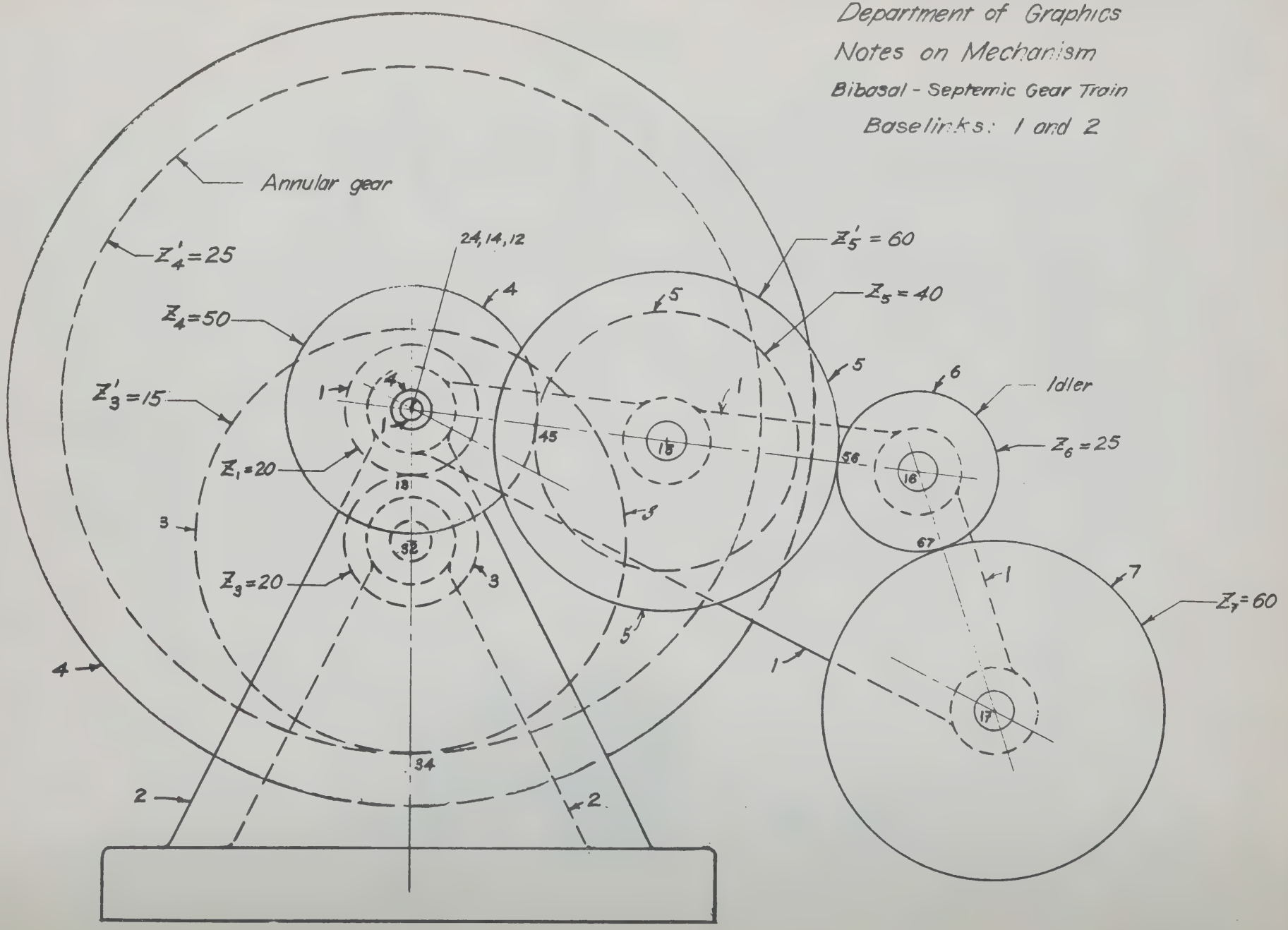

Fig. X-25

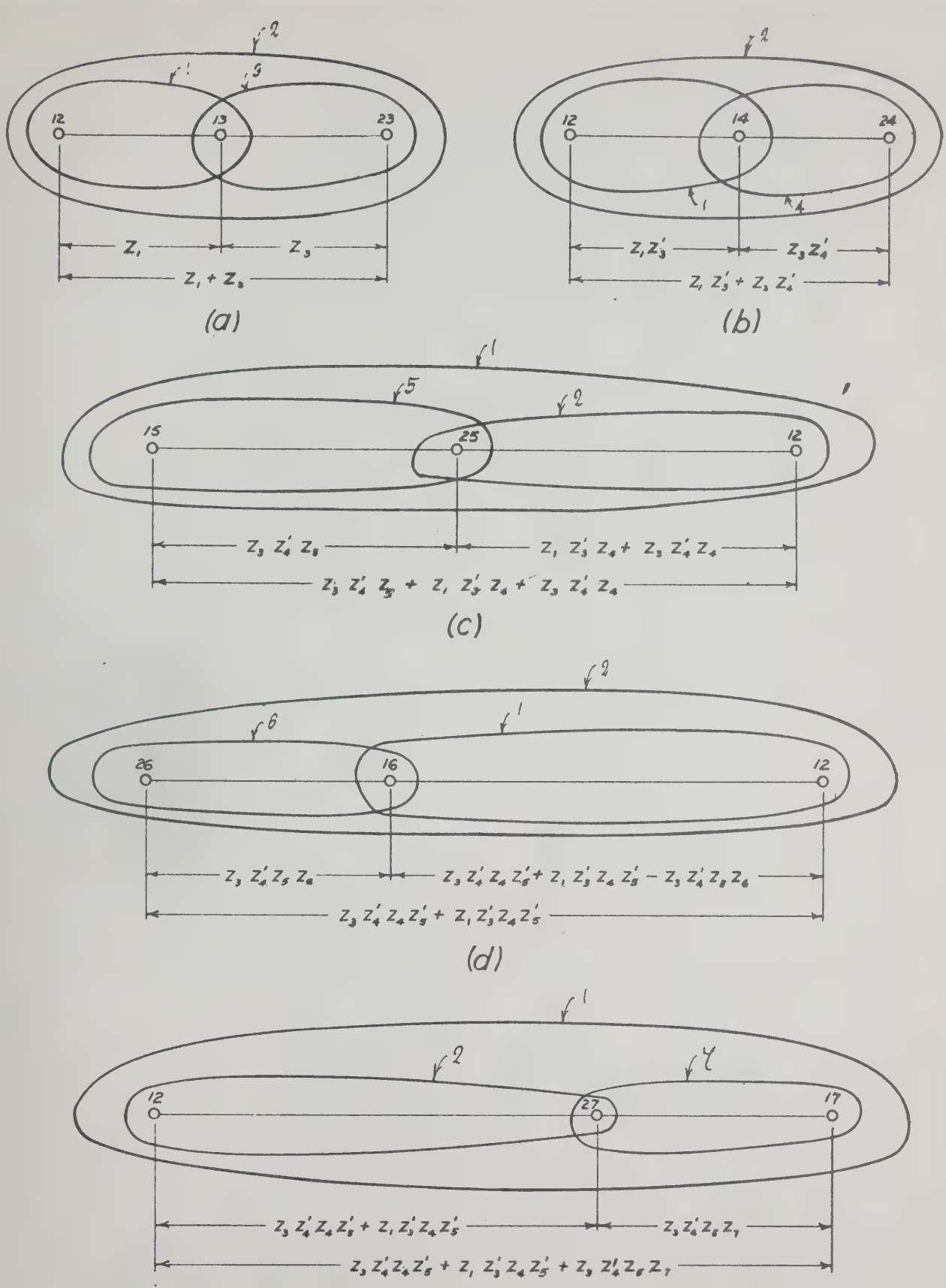
2
1
3
12
13
23
Z_1
Z_3
$Z_1 + Z_3$
(a)
2
12
14
24
1
4
$Z_1 Z_3'$
$Z_3 Z_4'$
$Z_1 Z_3' + Z_3 Z_4'$
(b)
1
5
2
15
25
12
$Z_3 Z_4' Z_5$
$Z_1 Z_3' Z_4 + Z_3 Z_4' Z_4$
$Z_3 Z_4' Z_5 + Z_1 Z_3' Z_4 + Z_3 Z_4' Z_4$
(c)
2
6
1
26
16
12
$Z_3 Z_4' Z_5 Z_6$
$Z_3 Z_4' Z_4 Z_5' + Z_1 Z_3' Z_4 Z_5' - Z_3 Z_4' Z_5 Z_6$
$Z_3 Z_4' Z_4 Z_5' + Z_1 Z_3' Z_4 Z_5'$
(d)
1
2
7
12
27
17
$Z_3 Z_4' Z_4 Z_5' + Z_1 Z_3' Z_4 Z_5'$
$Z_3 Z_4' Z_6 Z_7$
$Z_3 Z_4' Z_4 Z_5' + Z_1 Z_3' Z_4 Z_5' + Z_3 Z_4' Z_6 Z_7$
(e)

FIG. X'-25

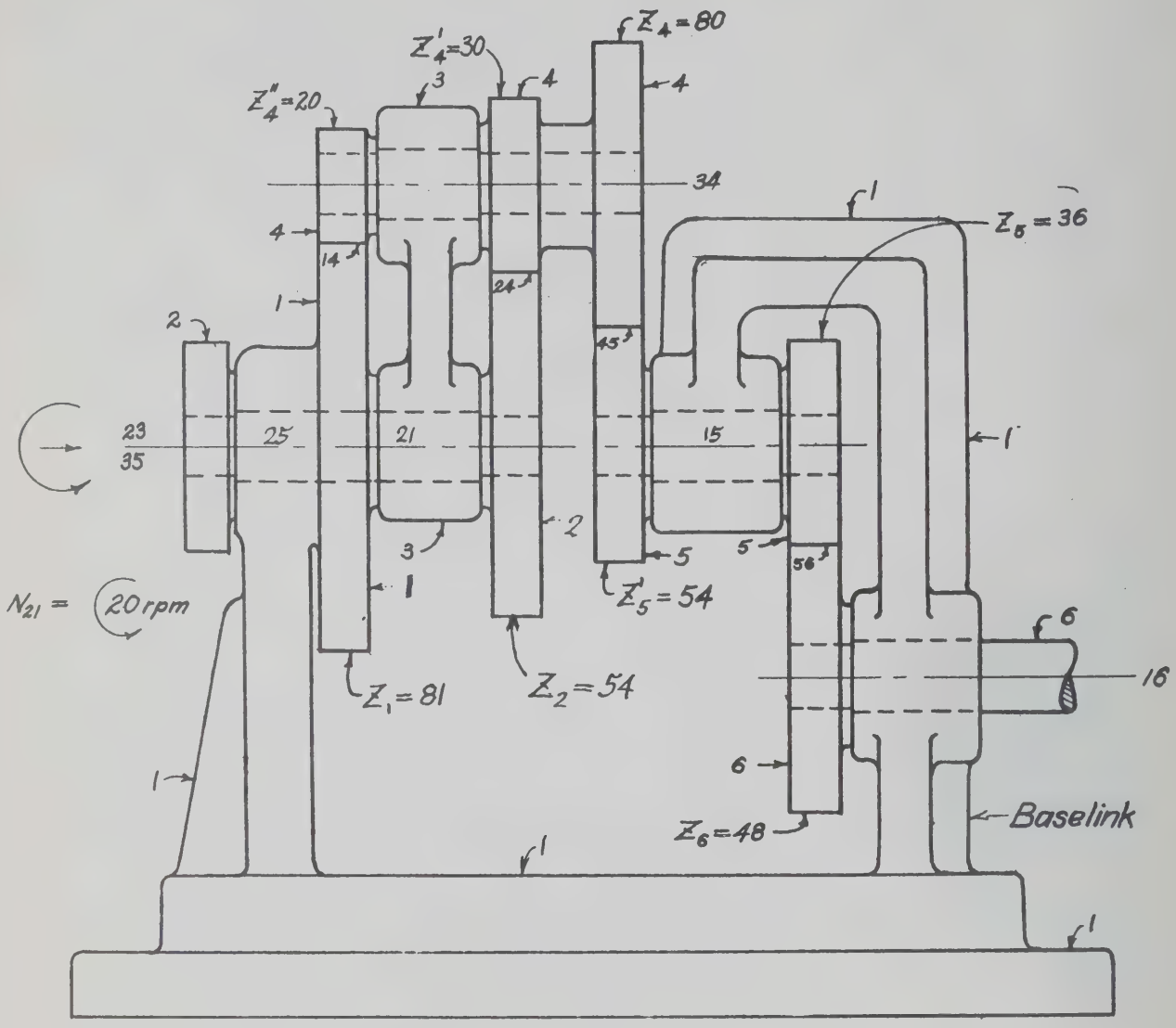

Fig. X-32

Bibasal Sextic Spur-gear Train.

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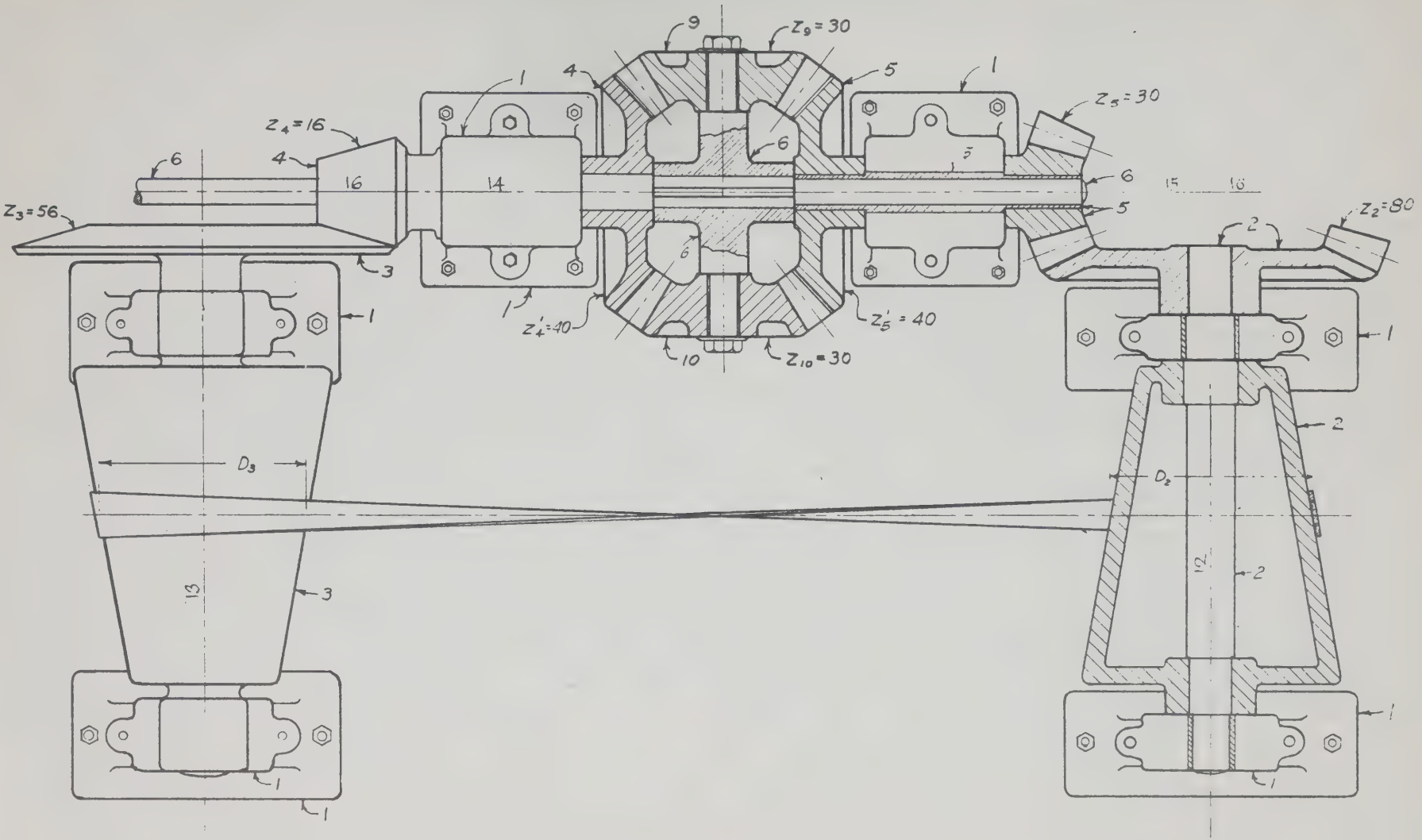
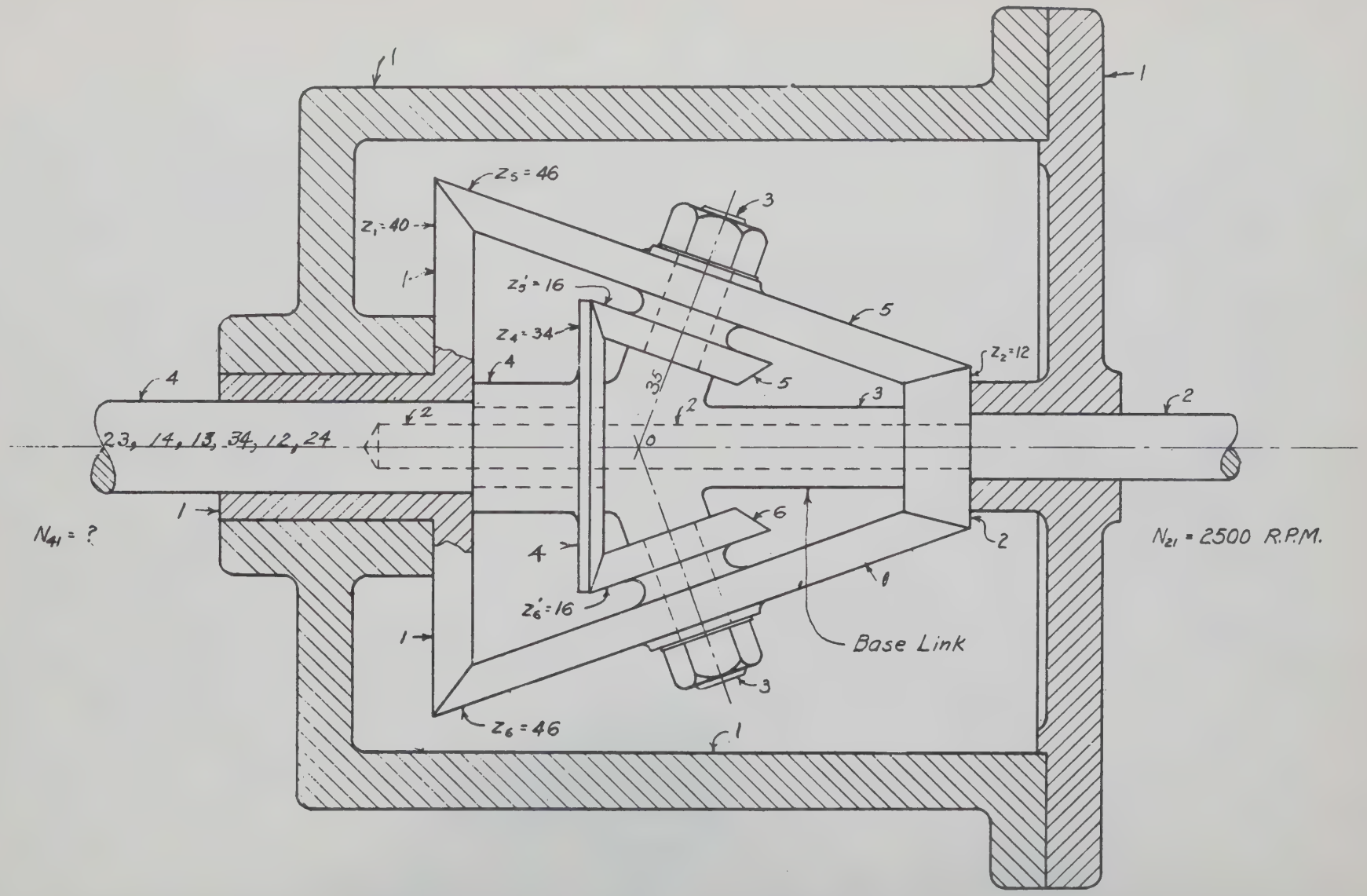
Notes on Mechanism



Water-Wheel-Governor Mechanism

FIG. X-34



MacCora's Unibasal Spheroquintic

FIG. X-39

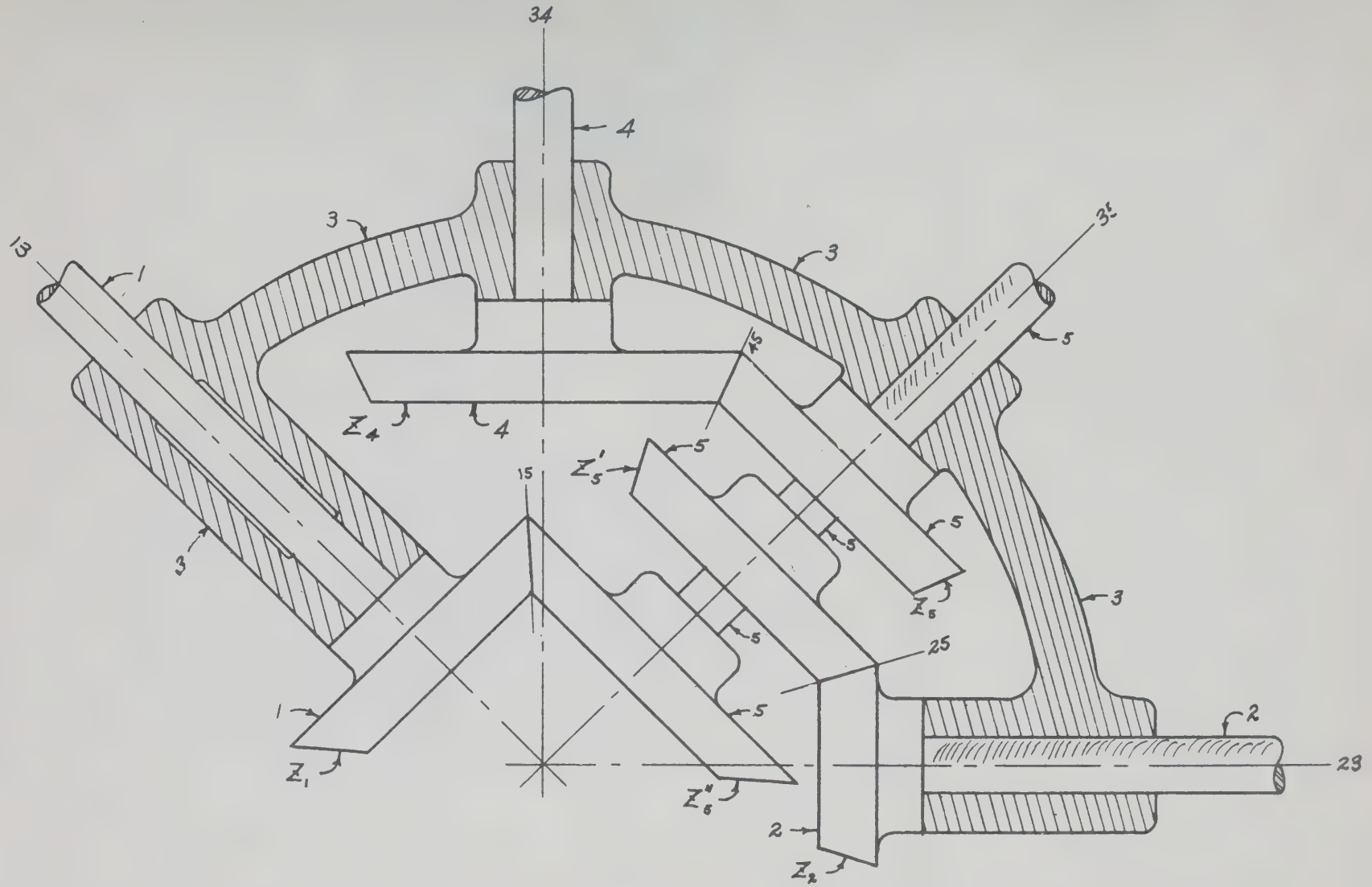


Fig. X-39,

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NOTES ON MECHANISM
GENERAL UNIBASAL SPHEROQUINTIC
Baselink 3

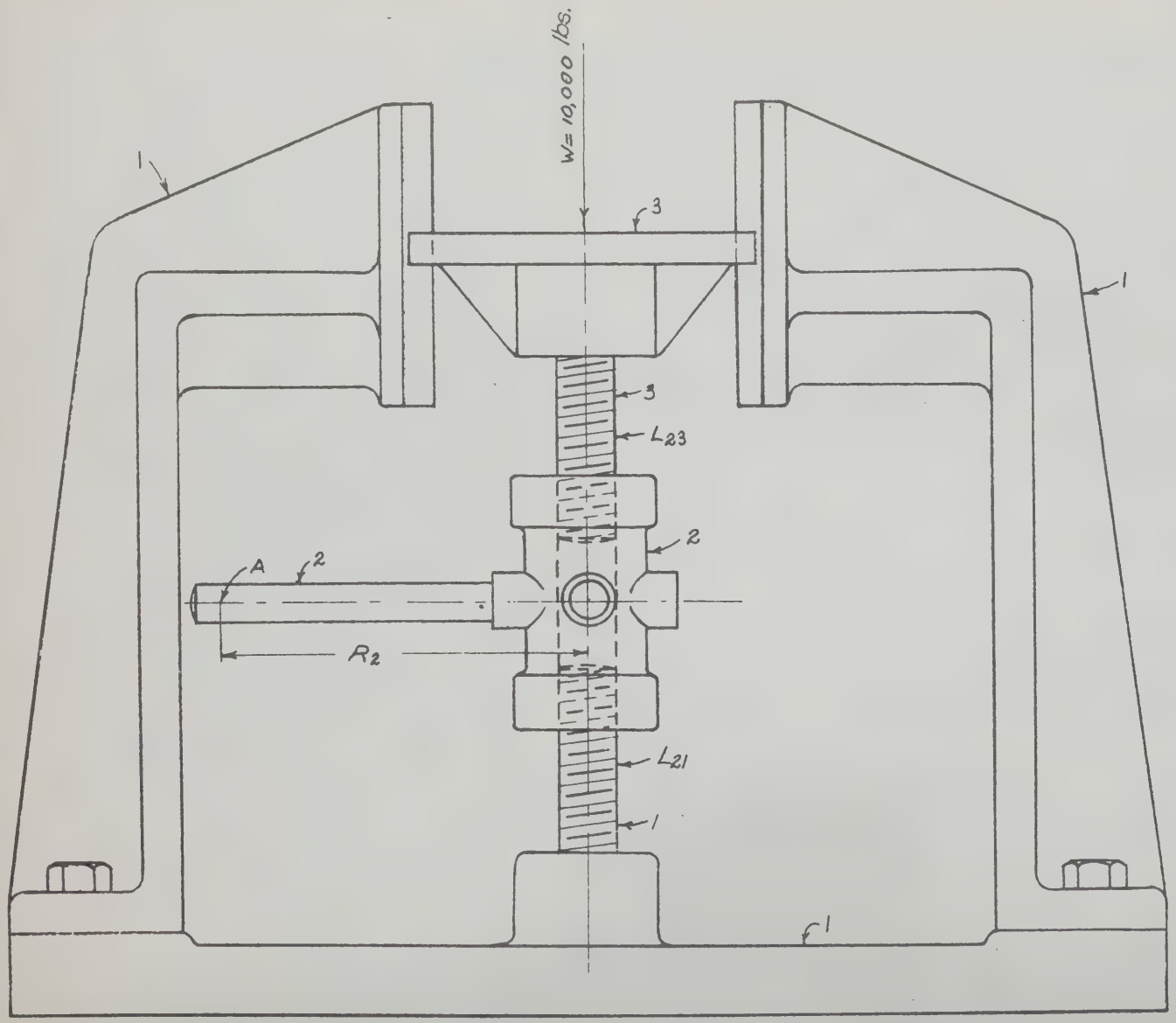

Fig. XII-10

Note: F_2 is applied at point A

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NOTES ON MECHANISM

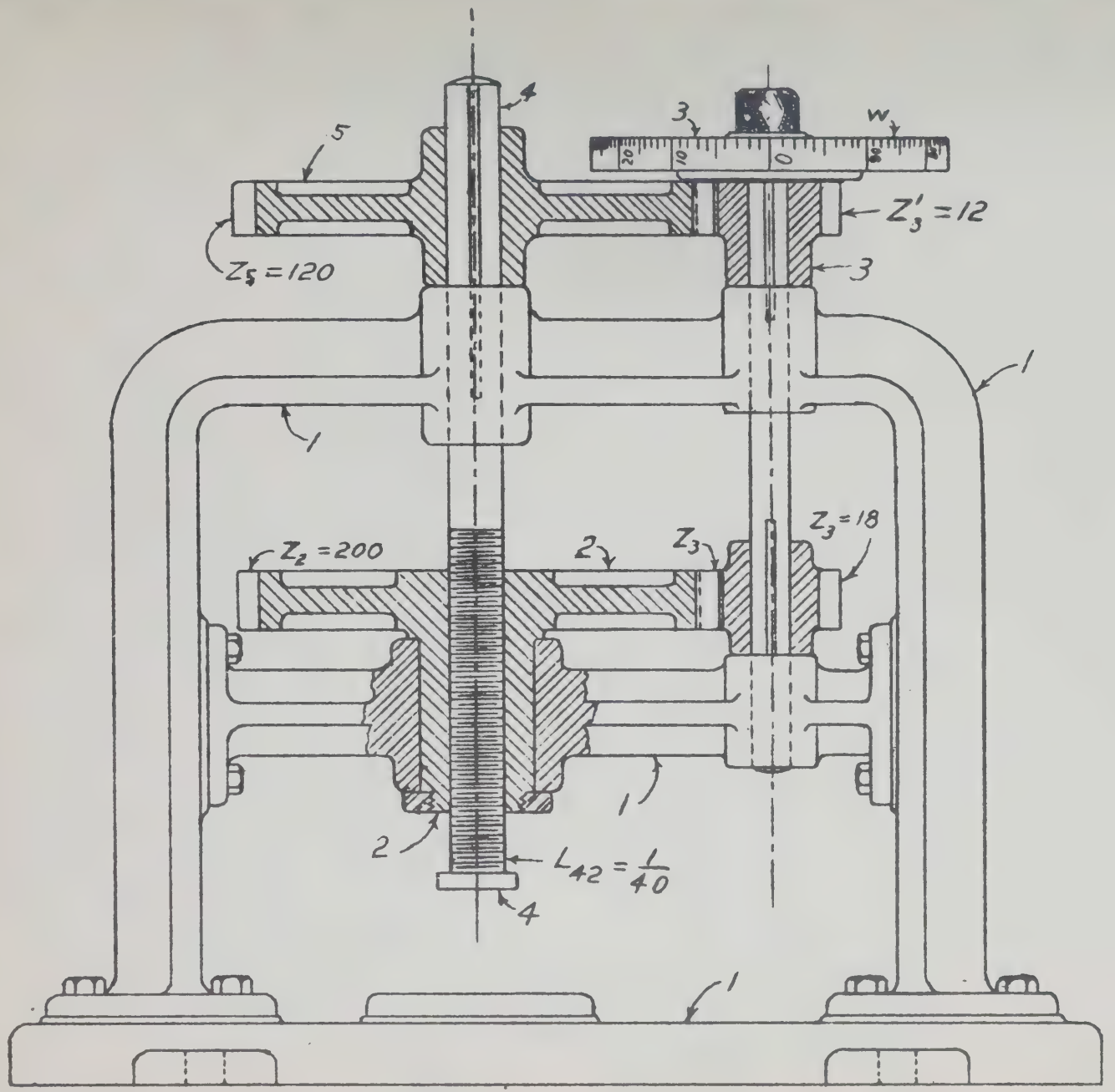

Fig. XII - 12

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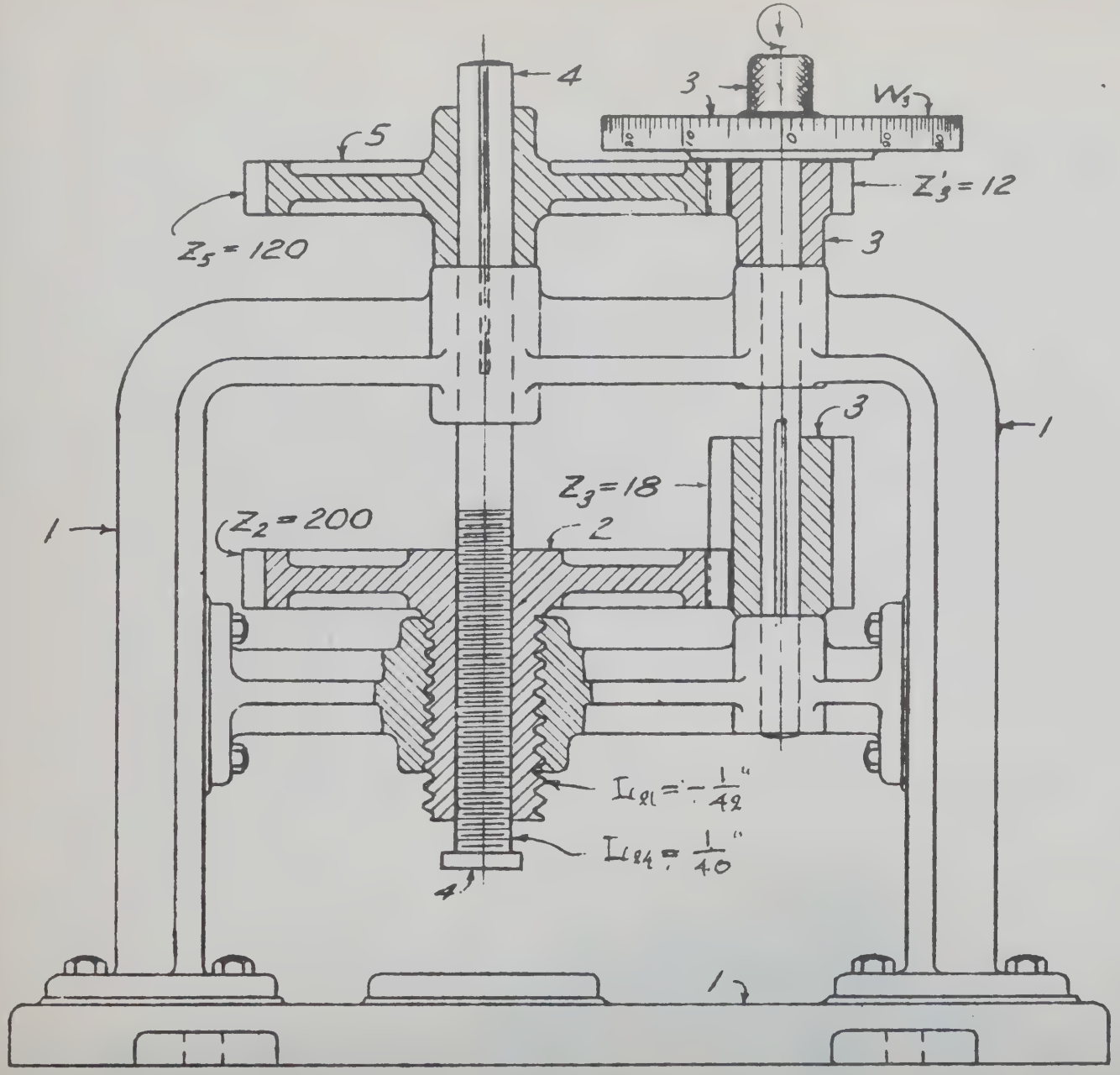
4
3
W_3
5
$Z'_3 = 12$
$Z_5 = 120$
3
3
1
$Z_3 = 18$
1
$Z_2 = 200$
2
$L_{21} = -\frac{1}{42}$"
$L_{24} = \frac{1}{40}$"
4
1

FIG. XII–13

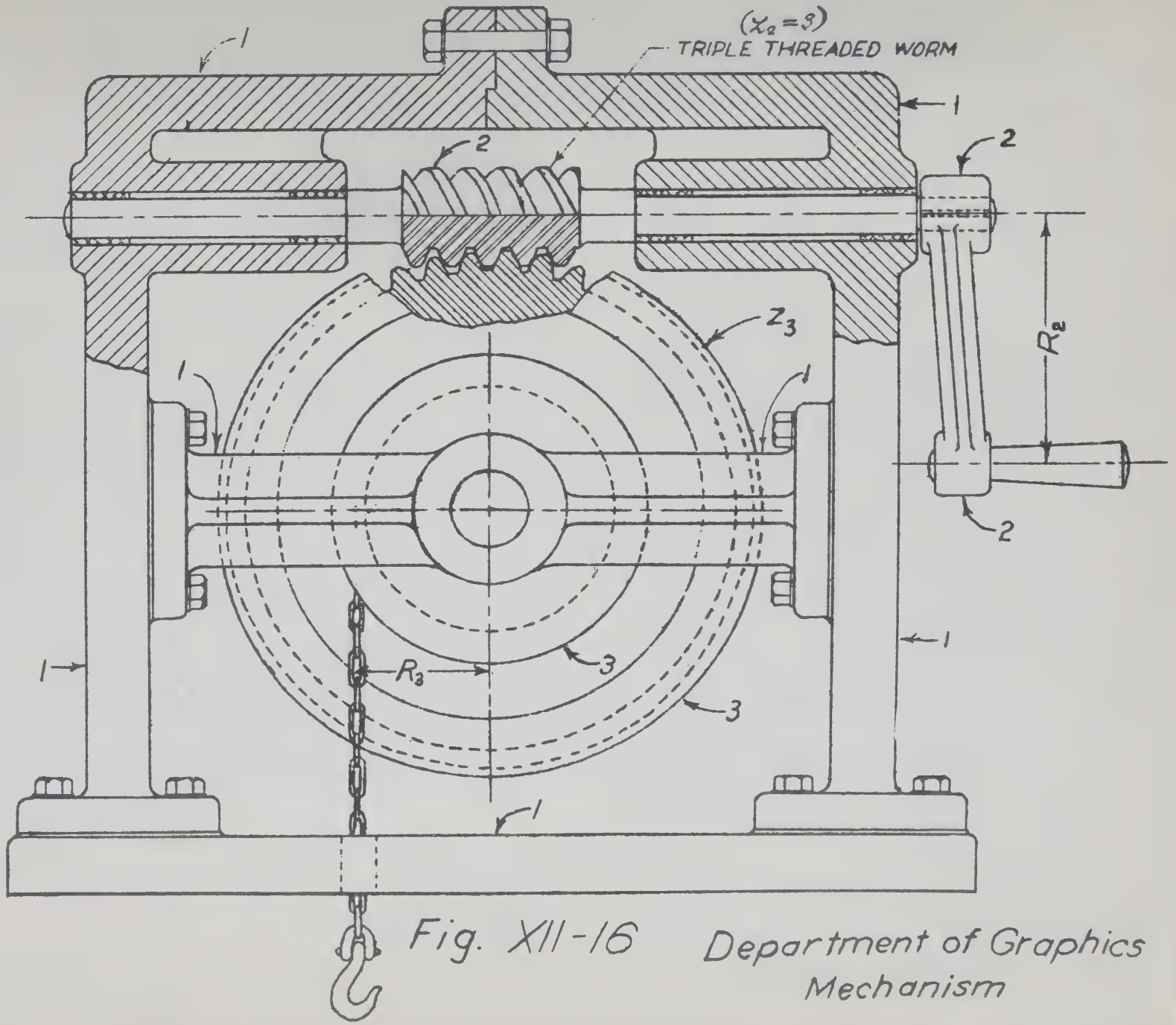

Fig. XII-16

Department of Graphics
Mechanism

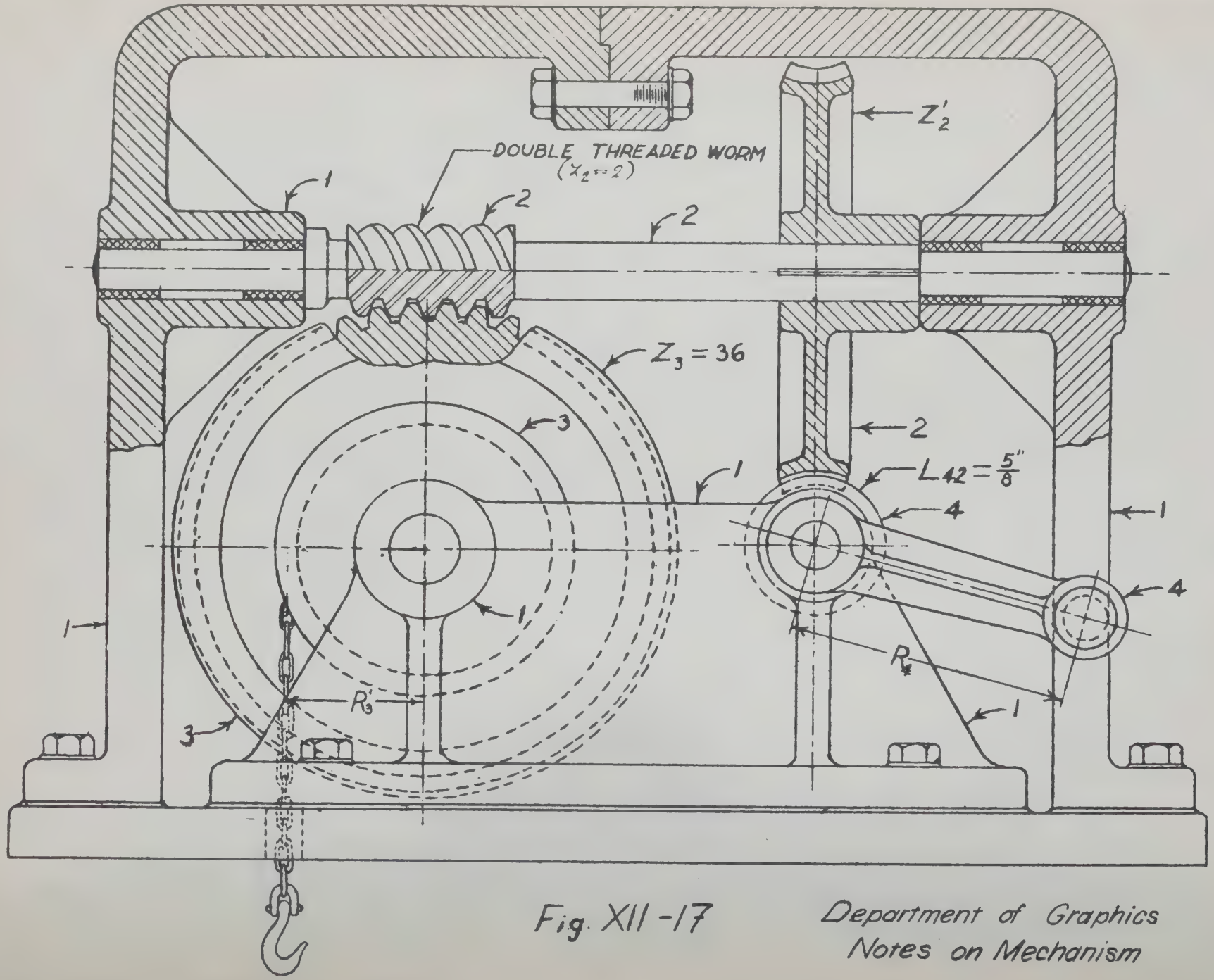

Fig. XII-17

Department of Graphics
Notes on Mechanism

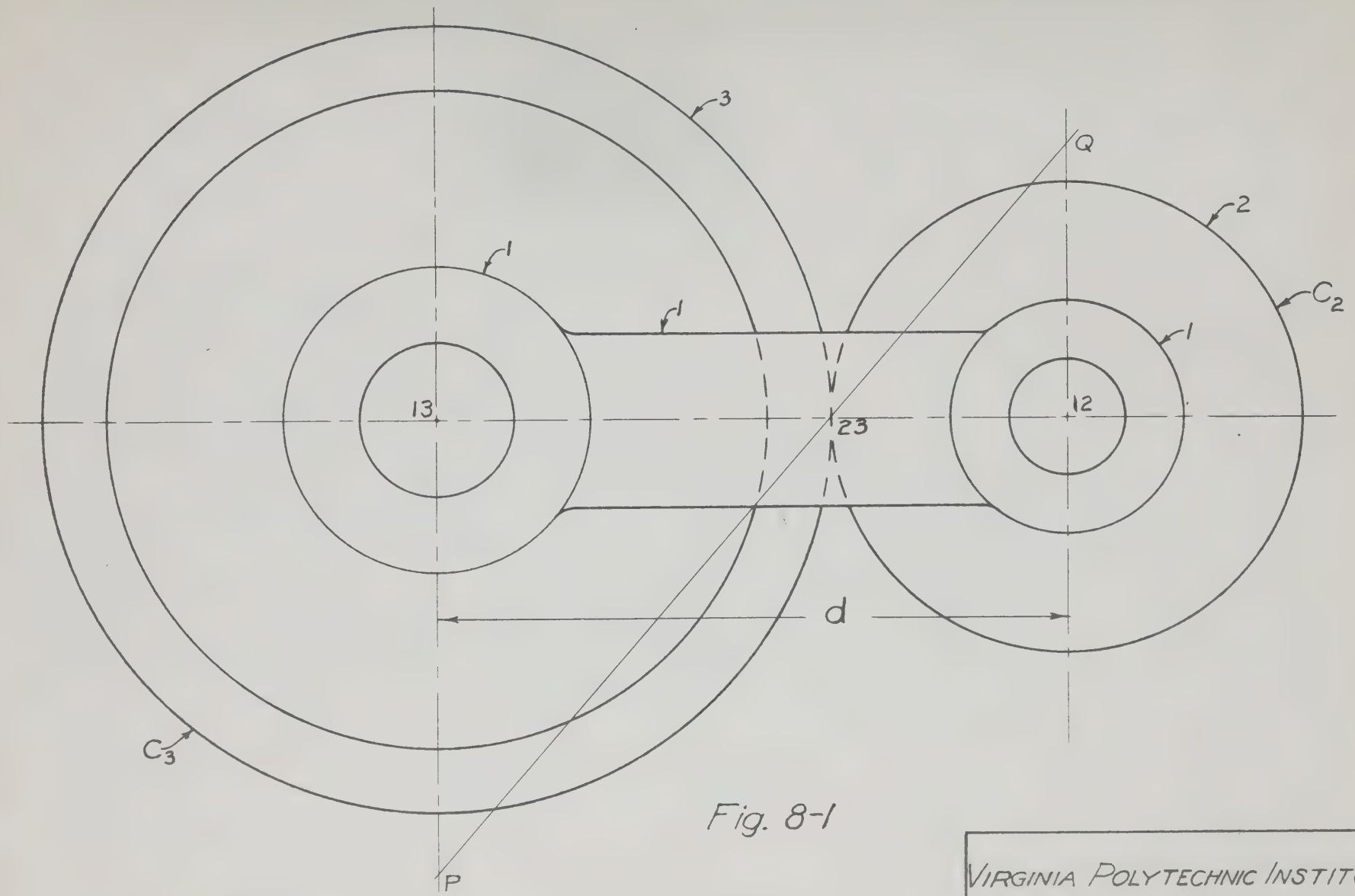

Fig. 8-1

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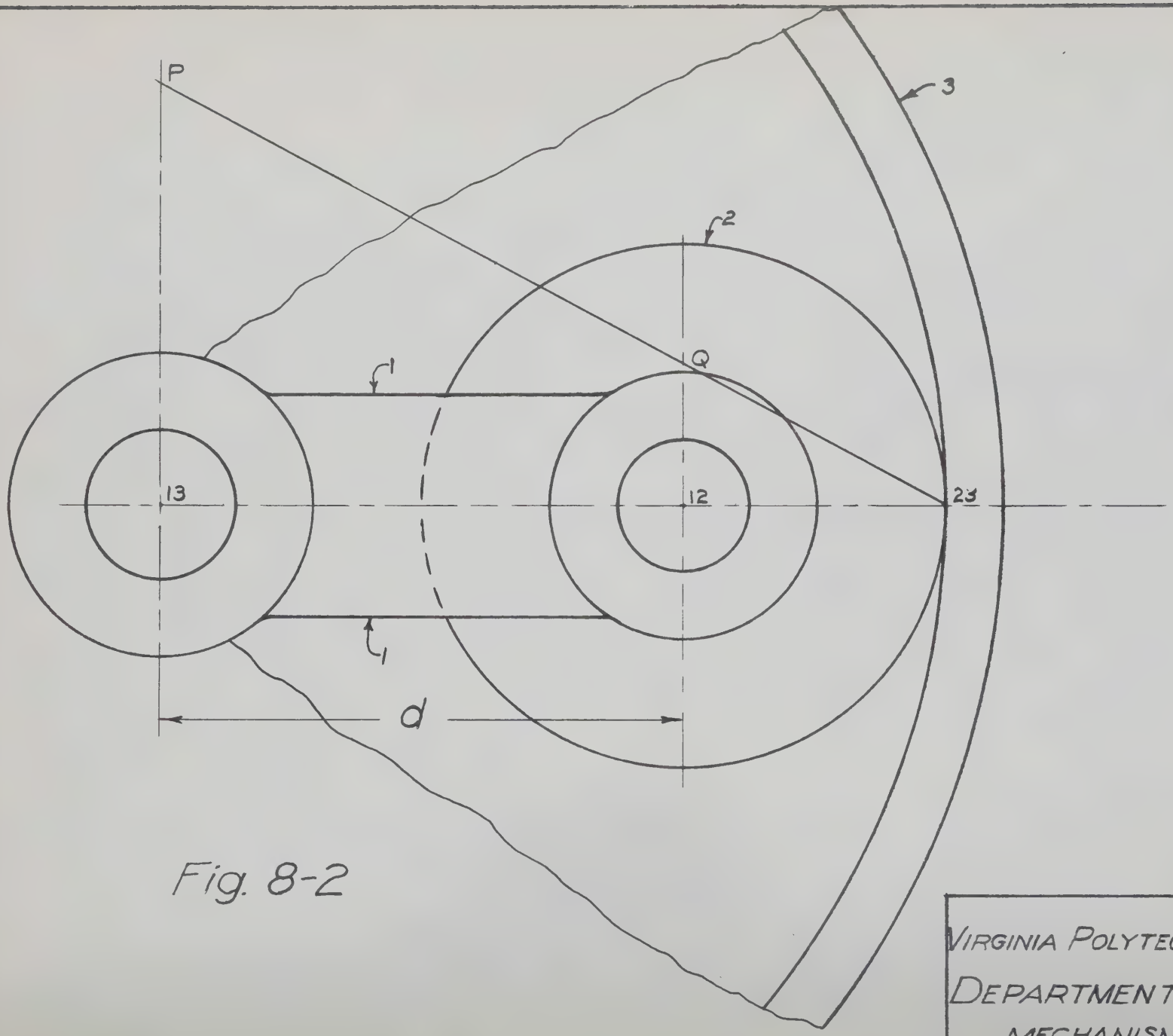

Fig. 8-2

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MECHANISM NOTES

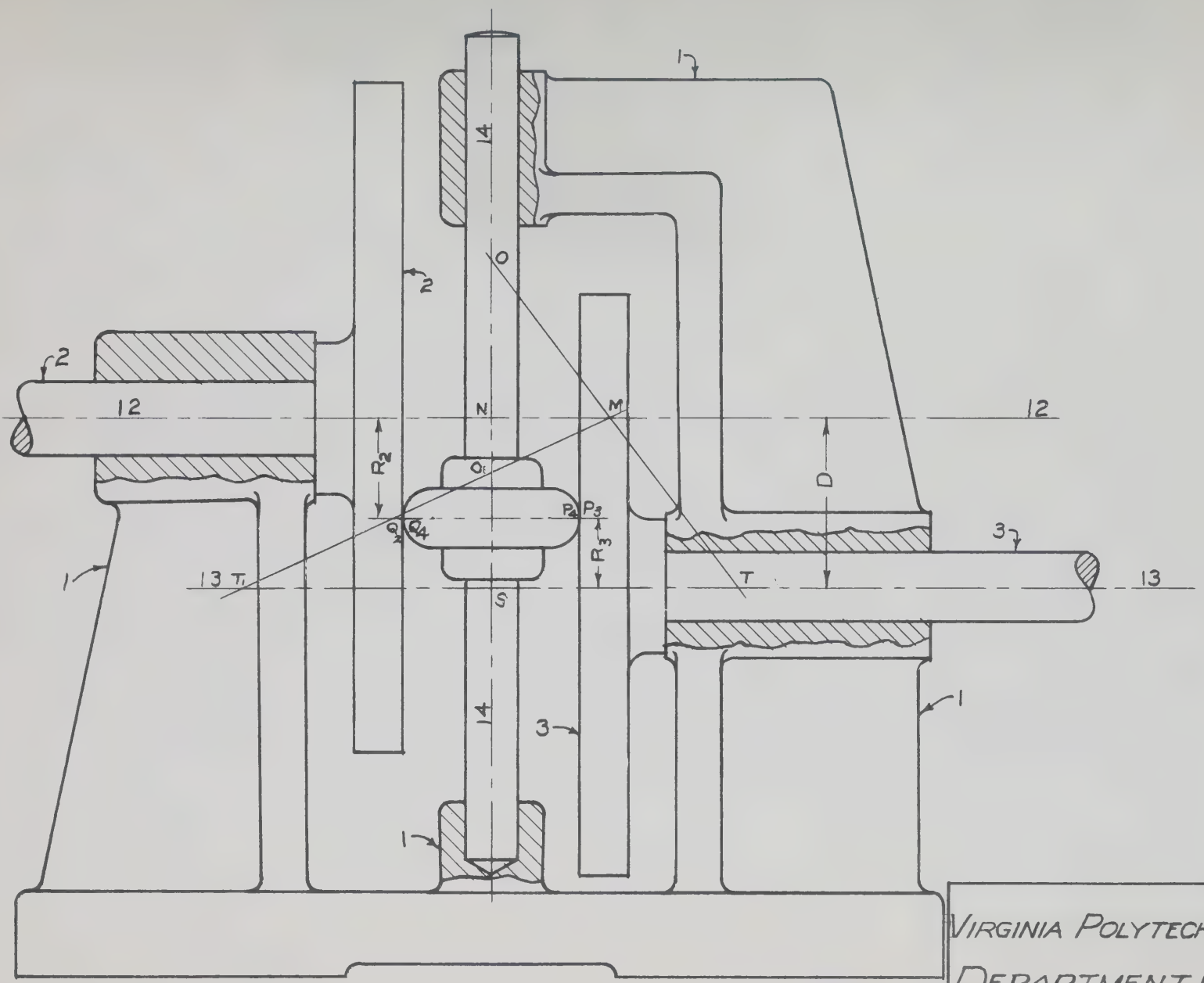

Fig. 8-2-3

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MECHANISM NOTES

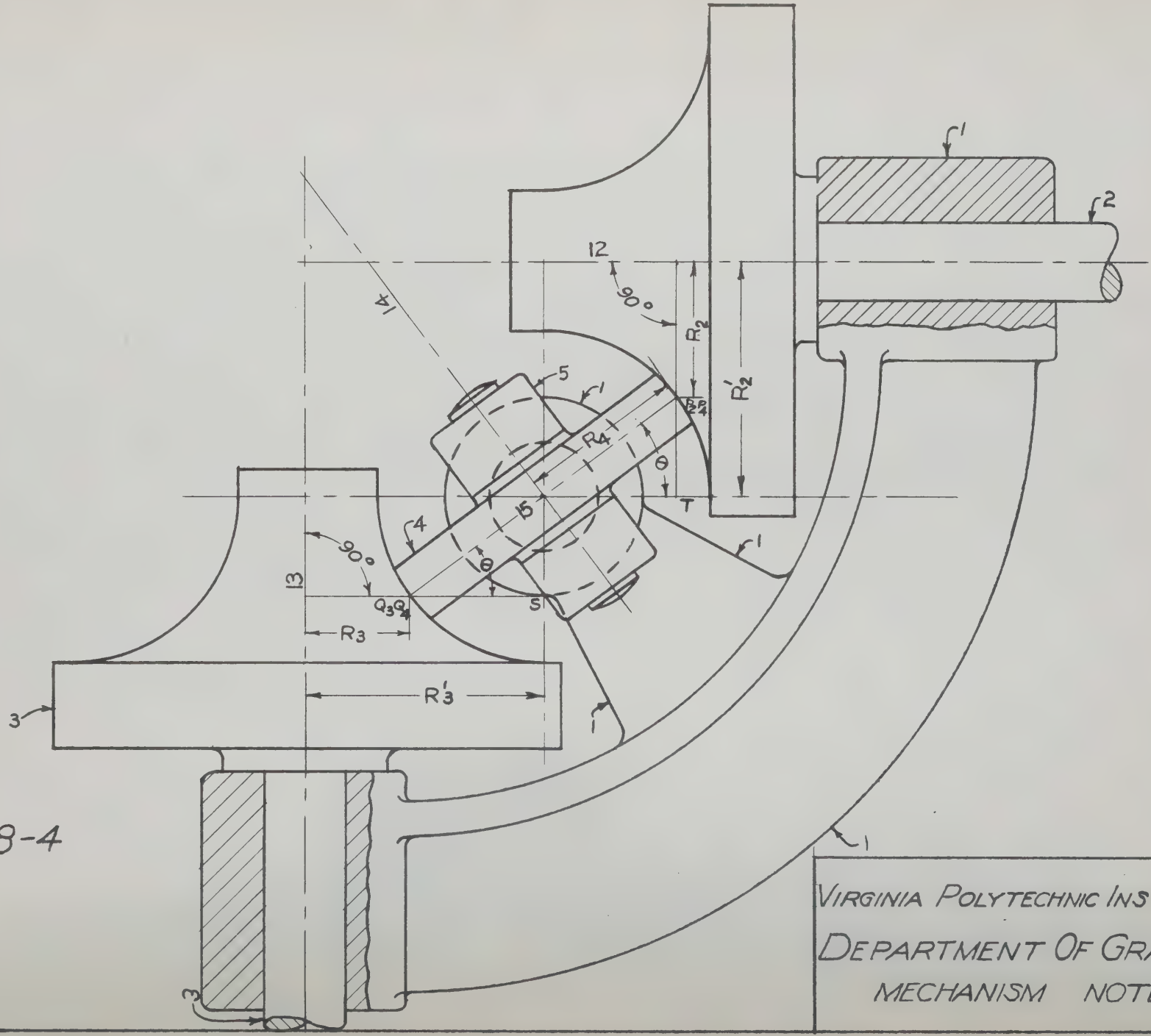

Fig. 8-4

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MECHANISM NOTES

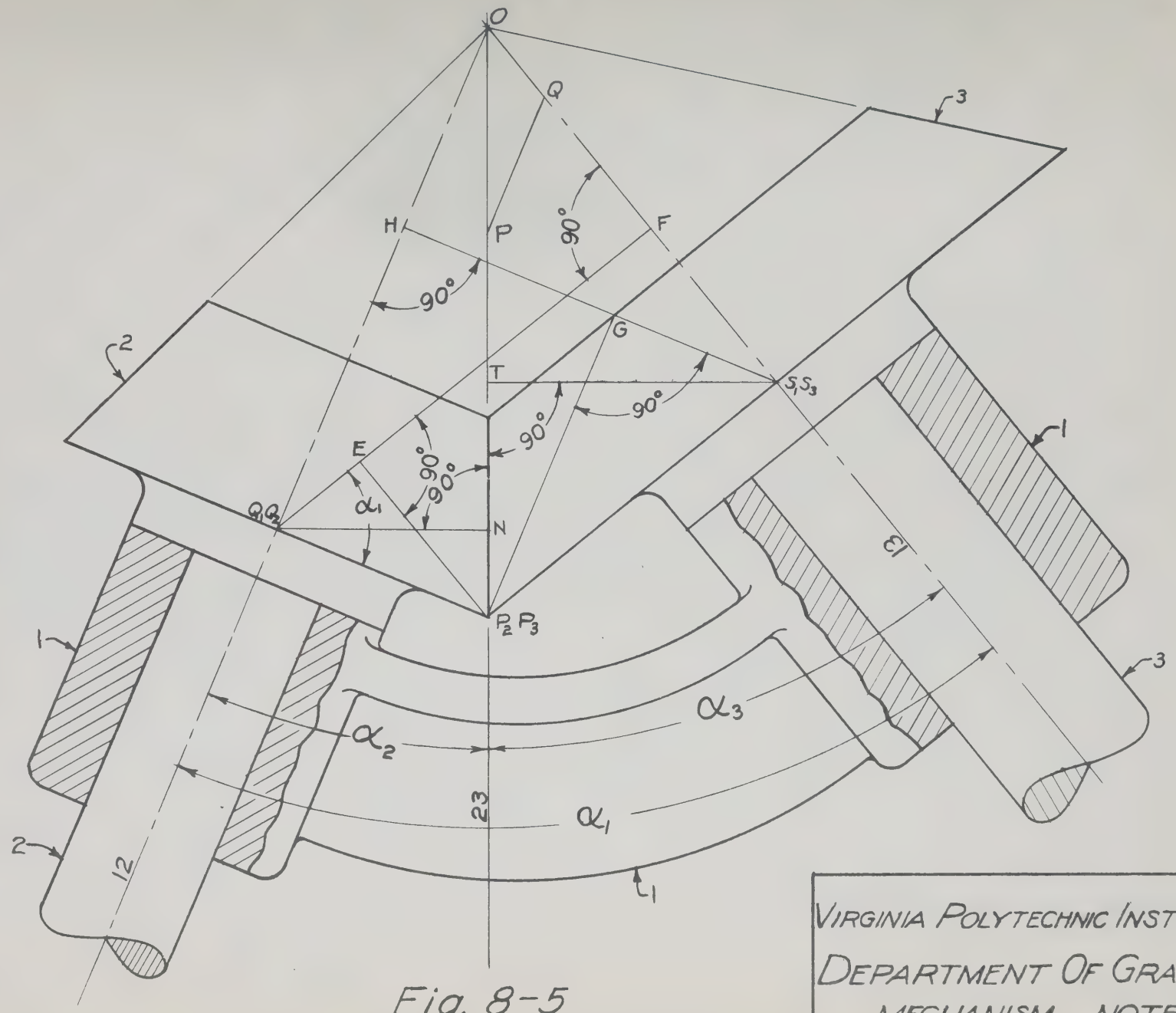

Fig. 8-5

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MECHANISM NOTES

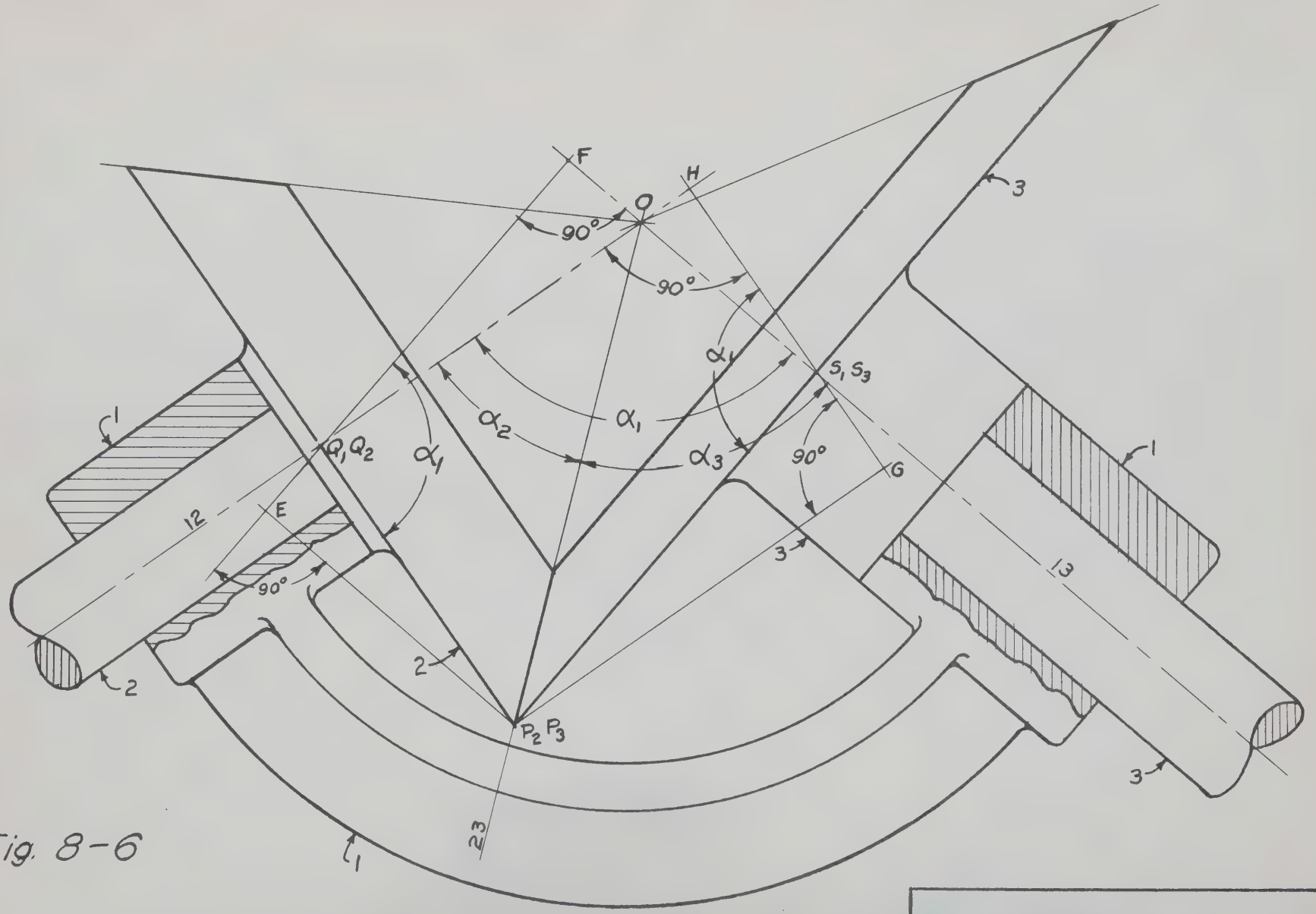

Fig. 8-6

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MECHANISM NOTES

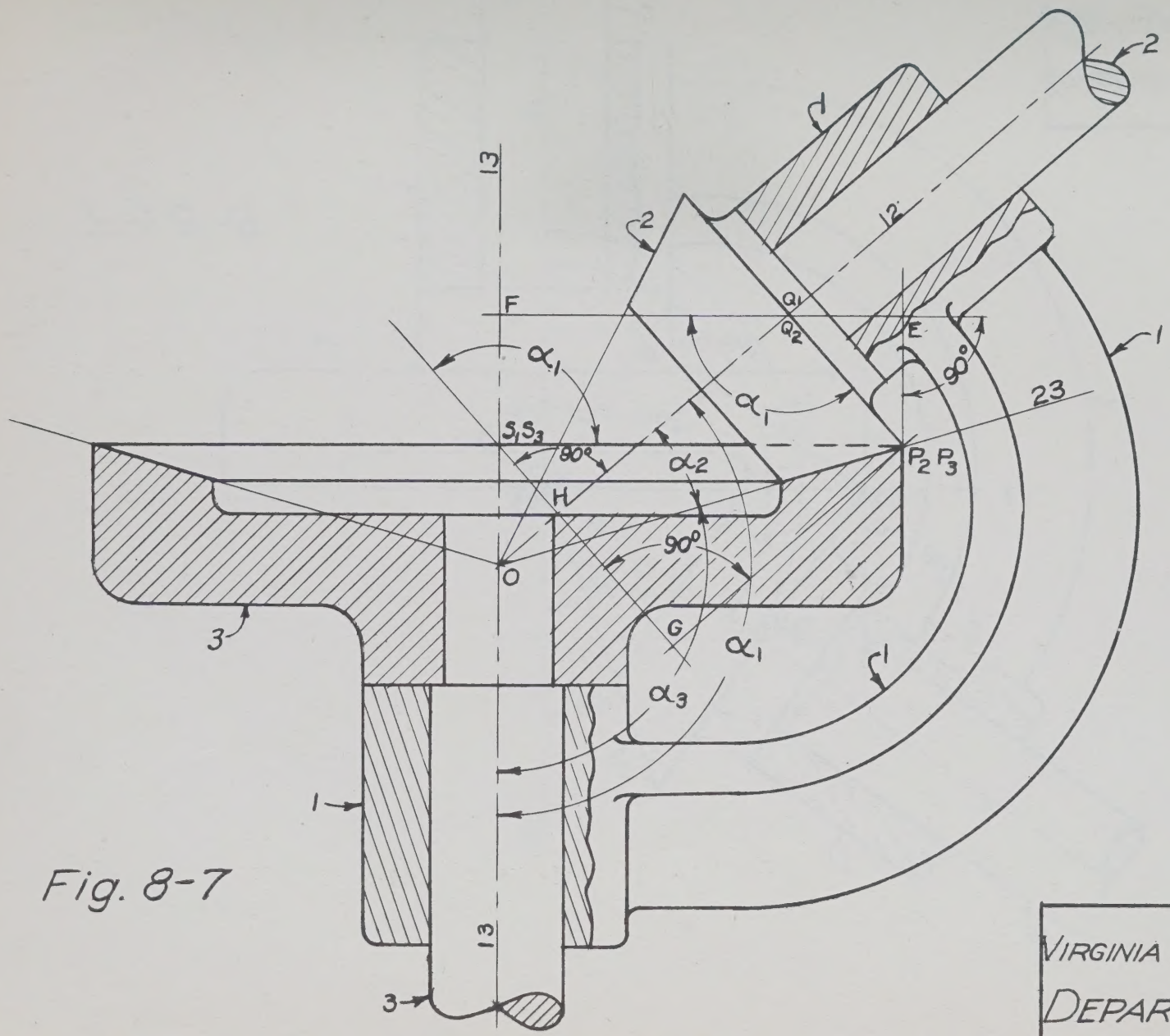

Fig. 8-7

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MECHANISM NOTES

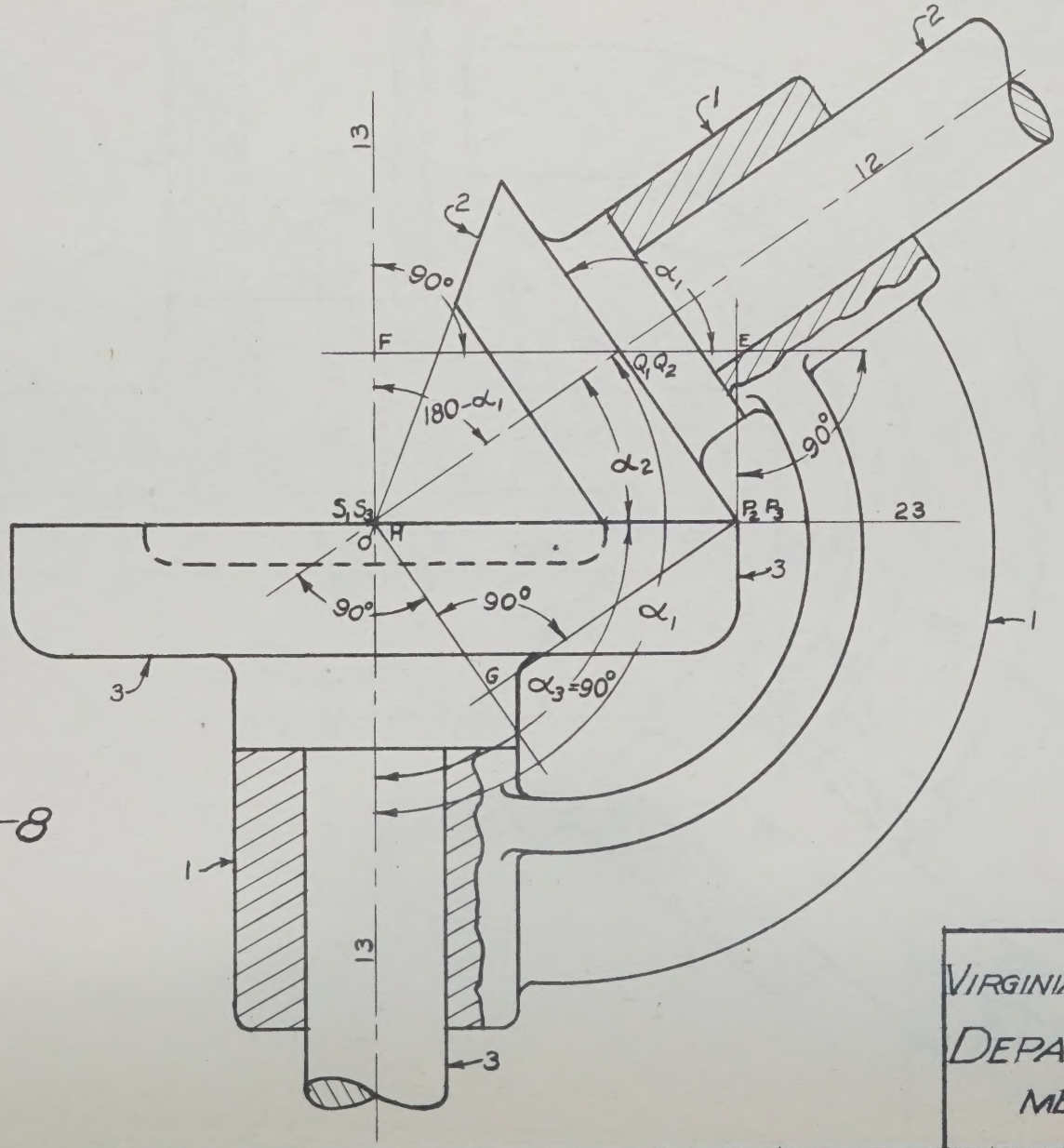

Fig. 8-8

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3 0112 077689765